Question 2.

(a) Define a sequence of positive integers $A_1, A_2, A_3\ldots$ as follows: $A_1 = 1$, and for $n \geq 2$ set

$$A_n = \sum_{j=1}^{n-1} A_j.$$

Prove, by induction, that $A_n = 2^{n-2}$ for all $n \geq 2$.

(b) In this question you can assume all standard facts about inqualities for real numbers.

Let X be the set of non-negative real numbers: $X = \{x \in \mathbf{R} : x \geq 0\}$. Let us investigate whether a version of strong induction works for X. So imagine that for every $x \in X$ we have a true/false statement $P(x)$, and let's say that we know

i) $P(0)$ is true; and

ii) for every $d \in X$, if $P(y)$ is true for all $0 \leq y < d$, then $P(d)$ is true.

Can we deduce that $P(x)$ is true for all $x \in X$?

Hint: try letting $P(x)$ be the statement "$x \leq 1$".

Note that I am expecting a careful proof, not just a one word answer consisting of a yes/no guess (for which you will get 0 marks, even if you get it right, because it's pretty obvious from the question that the answer is "no").

Answer.

(a) Note to markers: I have been doing induction super-carefully with statements $P(n)$ and showing $P(d)$ implies $P(d+1)$ etc, but I would be happy with a slightly more informal approach as long as the key points are there.

Note first that $A_2 = A_1 = 1$, $A_3 = A_1 + A_2 = 1 + 1 = 2$ and $A_4 = A_1 + A_2 + A_3 = 1 + 1 + 2 = 4$, so this does look good (although it doesn't give us any marks).

For $n \geq 2$, let $P(n)$ denote the statement that $A_n = 2^{n-1}$. Then we just checked above that $P(2)$ is true.

Now say $d \geq 2$ and $P(d)$ is true. Then we can deduce that $A_d = 2^{d-1}$. We also know that $A_d = A_1 + A_2 + \cdots + A_{d-1}$ (by definition). Hence

$$\begin{aligned} A_{d+1} &= A_1 + A_2 + \cdots + A_{d-1} + A_d \\ &= (A_1 + A_2 + \cdots + A_{d-1}) + A_d \\ &= A_d + A_d \\ &= 2A_d \\ &= 2.2^{d-1} = 2^d = 2^{(d+1)-1} \end{aligned}$$

and in particular we have just deduced $P(d+1)$ from $P(d)$.

So by the principle of mathematical induction (starting at $n = 2$) we can deduce $P(n)$ for all $n \geq 2$.

Note to markers: it's OK if the students use "strong induction"; if we assume $P(m)$ for all $m < n$ then $A_n = 1 + 1 + 2 + 4 + \cdots + 2^{n-2}$ which I think we can assume equals 2^{n-1} (although probably the way to prove this would be by induction ;-)). **(5 marks)**

(b) Note to markers: no marks for a one-word yes/no answer, even if they get it right.

Let us prove that if $P(x)$ is the statement "$x \leq 1$" then $P(x)$ satisfies the following three things:

i) $P(0)$ is true

ii) For every $d \in X$, if $P(y)$ is true for all $y < d$ then $P(d)$ is true.

iii) There does exist some $x \in X$ such that $P(x)$ is false.

In particular, strong induction does *not* work on the non-negative reals.

Two of these things are easy to check. First $P(0)$ is true, because $0 \leq 1$. Secondly, if we set $x = 2$ then $P(x)$ is false, because $2 > 1$. **(2 marks)** Note to markers: I am happy to give out two marks for everything so far. The guts of the question is what remains.

All we have to do then, is to prove that for every $d \in X$, if $P(y)$ is true for all $y < d$, then $P(d)$ is also true.

We split the argument into two cases. First, let us consider the case $d \leq 1$. Then $P(d)$ is true, so it's implied by anything at all (and in particular it's implied by the assertion that $P(y)$ is true for all $y < d$, whether or not this assertion is true, although it does happen to be true).

Finally, we need to check that if $d > 1$ and if $P(y)$ is true for all $y < d$, then $P(d)$ is true. Well, $P(d)$ is *not* true, so we need to check that our hypothesis of $P(y)$ being true for all $y < d$ is also false. And the trick, which goes back to a previous example sheet, is to set $y = (1+d)/2$, the average of 1 and d. Then $1 < y < d$, so in particular $y < d$ and $P(y)$ is false. **(3 marks)** Note to markers: I would be very happy to give marks to anyone who explains this in a coherent manner, even if they are twice as brief as I am. However I guess I would expect to see some sort of construction like $y = (1 + d)/2$ somewhere, for full marks (or some similar good idea).