Question 1. In this question you can assume any standard results (about binomial coefficients, the cosine function and so on).

(a) What is the prime factorization of the positive integer

$$1 - 3\binom{60}{2} + 9\binom{60}{4} - 27\binom{60}{6} + \cdots + 3^{30}\binom{60}{60}?$$

(b) Find real numbers α, β and γ so that the three roots of the polynomial $p(x) = 8x^3 - 6x - 1 = 0$ are $\cos(\alpha)$, $\cos(\beta)$ and $\cos(\gamma)$.

Answer.

(a) We saw in lectures that expanding $(1 + i)^n$ using polar coordinates and using the binomial theorem told us some strange facts about alternating sums of binomial coefficients, but this question has some extra powers of 3 in which we're going to have to accommodate somehow.

So instead of $(1+i)^n$ let's try $(1+it)^n$ with t some fixed real number. This expands out as $1+nit-\binom{n}{2}t^2-i\binom{n}{3}t^3+\binom{n}{4}t^4+\cdots$. The real part of this is $1-t^2\binom{n}{2}+t^4\binom{n}{4}-\cdots$, which seems to indicate that we should take $t = \sqrt{3}$ and $n = 60$.

Indeed, the binomial expansion of $(1+i\sqrt{3})^{60}$ has real part exactly the number which appears in the question.

Now using polar coordinates we see that we're interested in the real part of $(2e^{i\pi/3})^{60}$ which is just 2^{60}, which is hence the answer.**(5 marks)**

(b) We recognise from the exercise at the beginning of lecture 11 that $c = \cos(20°)$ satisfies $4c^3 - 3c = \cos(60°) = 1/2$, so c is a root of $8c^3 - 6c - 1 = 0$.

Now we just have to understand what is going on a bit better, to find the other roots. We know that if $x = \cos(\theta)$ then $4x^3 - 3x = \cos(3\theta)$, so what we really want is three values of θ such that $\cos(3\theta) = \cos(60°)$. Inspired by our trick for spotting all the nth roots of unity, given one of them, we notice that $\cos(60°) = \cos(60° + 360°) = \cos(60° + 2 \times 360°)$, so dividing these angles by 3 we could set $\alpha = 20°$, $\beta = 20° + 120° = 140°$ and $\gamma = 20° + 240° = 260°$. Certainly $\cos(\alpha)$, $\cos(\beta)$ and $\cos(\gamma)$ will all be roots of $8x^3 - 6x - 1$, as $\cos(3\alpha) = \cos(3\beta) = \cos(3\gamma) = 1/2$.**(4 marks)**

To finish the question, I feel like we need some sort of justification that these numbers are all distinct (because if they weren't then there could be other roots to the cubic). Here's one argument. $\cos(\alpha) > 0$ and $\cos(\beta)$, $\cos(\gamma) < 0$, so that's a good start. Because $\cos(\theta) = -\cos(\theta + 180°)$, we see $\cos(\gamma) = -\cos(80°)$ and $\cos(\beta) = -\cos(-40°) = -\cos(40°)$. But cos is strictly decreasing between 0 and 90 degrees, so $\cos(40°) \neq \cos(80°)$, and indeed we have convinced ourselves that $\cos(\alpha)$, $\cos(\beta)$ and $\cos(\gamma)$ are all distinct. I am prepared to give this one mark to anyone who even makes some sort of indication that they are aware that this needs to be checked though (as well, of course, to anyone who comes up with a different argument for which the fact that we have all the roots comes out in the wash somehow).**(1 mark)**