**Question 4.**

(a) Are the following statements true or false? Proof or counterexample required.

i) $\forall x \in \mathbf{R}\, \forall y \in \mathbf{R}\; x + y = 2$

ii) $\exists x \in \mathbf{R}\, \exists y \in \mathbf{R}\; x + y = 2$

(b) I had some polynomial $p(x)$ with real coefficients written on some piece of paper somewhere, and I was going to ask you to find the set of all $x$ such that $p(x) < 0$. But I think I left it on the tube. I can remember the answer though, it was that $x$ had to be in the set $(1, 2) \cup (3, 4)$. Write down a polynomial $p(x)$ such that $p(x) < 0$ if and only if $x$ is in this set, and *briefly* (i.e. "this last bit is only worth one or two marks, and you can assume all reasonable facts about inequalities") justify why the answer is $(1, 2) \cup (3, 4)$.

**Answer.**

(a) i) This is not true. For a counterexample, we could take $x = 3$ and $y = 4$; then $x + y = 7$ and $7 \neq 2$. **(2 marks)**

ii) This one is true. To prove it we need to show that there exist two real numbers $x$ and $y$ such that $x+y = 2$. We could take $x = 0$ and $y = 2$ for example, giving $x + y = 0 + 2 = 2$, but of course there are infinitely many other possibilities. **(2 marks)**

(b) The polynomial $p(x) = (x - 1)(x - 2)(x - 3)(x - 4)$ works (as does any positive real multiple of this, as does $(x - 1)^3(x - 2)^7(3 - x)(4 - x)(x - 5)^2$ etc etc).**(4 marks)**

The reason it works is as follows. If $x = 1, 2, 3$ or 4 then $p(x) = 0$ which is not negative. For $x < 1$ all the terms are negative so the product is positive. For $1 < x < 2$ (resp. $2 < x < 3$, $3 < x < 4$, $x > 4$) we see that three terms (resp. 2, 1, 0 terms) are negative and the rest are positive, so the product is negative (resp. positive, negative, positive); hence the only times $p(x)$ is negative are when $1 < x < 2$ or $3 < x < 4$. **(2 marks)**