Question 1.

(a) Let S be a set, and let $\sim$ be a binary relation on S. What does it mean to say that $\sim$ is:

i) *reflexive*;

ii) *symmetric*;

iii) *transitive*;

iv) an *equivalence relation*?

(b) In this question there are going to be two binary relations, one on a set S and one on a different set T, and I don't want to use $\sim$ for both of them, so I need to use a new symbol, and it's going to be $\bowtie$.

So say S and T are sets, and $f : S \to T$ is a function, and $\bowtie$ is an equivalence relation on T (so if $t_1, t_2 \in T$ then $t_1 \bowtie t_2$ is either true or false, and $\bowtie$ satisfies all the axioms for an equivalence relation). Define a binary relation $\sim$ on S by, for all $s_1, s_2 \in S$,

$$s_1 \sim s_2 \iff f(s_1) \bowtie f(s_2).$$

Prove that $\sim$ is an equivalence relation on S.

(c) How many binary relations are there on a set with two elements? How many of these binary relations are reflexive?

Answer.

(a) These are all definitions so it's traditional to write "if", but I'm of course not bothered if people write "iff".

i) $\sim$ is reflexive if $\forall s \in S\ s \sim s$. **(1 mark)**

ii) $\sim$ is symmetric if $\forall s, t \in S$, if $s \sim t$ then $t \sim s$. **(1 mark)**

iii) $\sim$ is transitive if for all $s, t, u \in S$, if $s \sim t$ and $t \sim u$ then $s \sim u$. **(1 mark)**

iv) $\sim$ is an equivalence relation if it's reflexive, symmetric and transitive. **(1 mark)**

(b) Certainly $\sim$ is a binary relation, because $\bowtie$ is. Now let's check the axioms.

If $s \in S$ then $f(s) = f(s)$, and $\bowtie$ is reflexive, so $f(s) \bowtie f(s)$, so $s \sim s$. Hence $\sim$ is reflexive. **(1 mark)**

If $s, t \in S$ and $s \sim t$, then by definition $f(s) \bowtie f(t)$. But $\bowtie$ is symmetric, hence $f(t) \bowtie f(s)$, and hence $t \sim s$. Hence $\sim$ is symmetric. **(1 mark)**

Finally, If $s, t, u \in S$ and $s \sim t$ and $t \sim u$, then by definition $f(s) \bowtie f(t)$ and $f(t) \bowtie f(u)$. But $\bowtie$ is transitive, hence $f(s) \bowtie f(u)$, and hence by definition $s \sim u$. Hence $\sim$ is transitive. **(1 mark)**

So $\sim$ is reflexive, symmetric and transitive, and thus it's an equivalence relation.

(c) A binary relation can be thought of as a subset of $S \times S$, and $S \times S$ has four elements; each of these elements is either in a subset or not, giving two choices for each element, so there are a total of $2^4 = 16$ binary relations on S. **(1 mark)**

Let's say $S = \{s, t\}$. For a general binary relation we would have to decide whether the following four things are true or false: $s \sim s$, $s \sim t$, $t \sim s$ and $t \sim t$. But if we know that $\sim$ is reflexive then we know that $s \sim s$ and $t \sim t$ must be true, so we only need to decide about whether $s \sim t$ and whether $t \sim s$. This is two true/false choices and we can hance make those decisions in $2 \times 2 = 4$ ways, so there are four reflexive binary relations on a set with two elements. **(2 marks)**