Question 1.

Let us say that a function $f\colon (0,1) \to \mathbb{R}$ is *increasing* if for all $t_1, t_2 \in (0,1)$, $t_1 < t_2$ implies $f(t_1) < f(t_2)$.

Let $f\colon (0,1) \to \mathbb{R}$ be an increasing function, and fix $t_0 \in (0,1)$. Define subsets S, $T \subset \mathbb{R}$ as follows:

$$S = \{f(t) \mid t < t_0\}, \quad \text{and} \quad T = \{f(t) \mid t > t_0\}$$

Show that S is nonempty and bounded above; that T is nonempty and bounded below, and that

$$\sup S \leq \inf T$$

but that neither is necessarily equal to $f(t_0)$.

Answer.

It is clear that S and T are nonempty: if $t^\dagger < t_0$ then $f(t^\dagger) \in S$ and if $t^\star > t_0$ then $f(t^\star) \in T$.

S is bounded above by $f(t^\star)$: if $s \in S$ then $s = f(t')$ for some $t' < t_0$ but then also $t' < t^\star$ so $s = f(t') < f(t^\star)$.

Similarly, T is bounded below by $f(t^\dagger)$. **(3 marks)**

Let $a = \inf T$. Fix $\varepsilon > 0$. From the Lemma proved in the lectures, we know that there is some $t_0 < t_+$ with $f(t_+) < a + \varepsilon$. Then for all $t' < t_0$ we have $f(t') < f(t_+) < a + \varepsilon$. Thus we see that for all $\varepsilon > 0$, $a + \varepsilon$ is an upper bound for S.

So for all $\varepsilon > 0$, $\sup S \leq a + \varepsilon$. This implies $\sup S \leq a$. **(3 marks)**

Let now $f\colon (0,1) \to \mathbb{R}$ be defined as follows:

$$f(t) = \begin{cases} t & \text{if } t \leq \frac{1}{2} \\ t + \frac{1}{2} & \text{if } t > \frac{1}{2} \end{cases}$$

then f is increasing. If we set $t_0 = \frac{1}{2}$, then $\sup S = \frac{1}{2}$ and $\inf T = 1$. **(4 marks)**