Question 2.

(a) Find all $x, y \in \mathbb{Z}$ that satisfy the equation:

$$x^2 \equiv 3y^2 \mod 7\,.$$

(b) Using part (a), find all $x, y, z \in \mathbb{Z}$ that satisfy the equation:

$$x^2 + 7y^2 = 3z^2\,.$$

Answer.

(a) **(4 marks)** All solutions are $x = 7p$, $y = 7q$ for $p, q \in \mathbb{Z}$. Indeed if $y \equiv 0 \mod 7$, then also $x \equiv 0 \mod 7$, and this leads to the stated solution. We show that $y \not\equiv 0 \mod 7$ is impossible. Indeed, in that case, $\text{hcf}(y, 7) = 1$ so, as shown in class, there is $u \in \mathbb{Z}$ with $yu \equiv 1 \mod 7$ and then $(xu)^2 \equiv 3 \mod 7$. This is impossible because 3 is not a square mod 7: $(\pm 1)^2 \equiv 1$, $(\pm 2)^2 \equiv 4$ and $(\pm 3)^2 \equiv 2 \mod 7$ so the squares mod 7 are $0, 1, 2$ and 4.

(b) **(6 marks)** The only solution is the trivial solution $x = 0, y = 0, z = 0$. Indeed, suppose for a contradiction that x, y, z is a nontrivial solution. Dividing through by the hcf we may assume that $\text{hcf}(x, y, z) = 1$. Reducing mod 7 we get that

$$x^2 \equiv 3z^2 \mod 7$$

so by part (a) there are integers x_0 and z_0 such that $x = 7x_0$ and $z = 7z_0$. Plugging into the original equation we get:

$$49x_0^2 + 7y^2 = 3 \times 49z_0^2$$

and, dividing through by 7, we get:

$$y^2 = 7(3z_0^2 - x_0^2)$$

so y is also divisible by 7, contradicting $\text{hcf}(x, y, z) = 1$.