Question 3.

In this question:

We say that an integer $a > 0$ is the sum of two integer squares if there are integers n, m such that $a = m^2 + n^2$;

For all real numbers a, b, $(a, b) = \{t \in \mathbb{R} \mid a < t < b\}$.

For each of the following statements, find a proof or a counterexample.

(a) If A, B and C are sets, then $A \cap (B \cup C) = (A \cap B) \cup C$.

(b) For all $a > 0$, $b > 0$ integers, ab is the sum of two integer squares if both a and b are sums of two integer squares.

(c) For all $a > 0$, $b > 0$ integers, ab is the sum of two integer squares only if both a and b are sums of two integer squares.

(d) For all $\varepsilon > 0$, $(-\varepsilon, \varepsilon) \setminus \mathbb{Q} \neq \emptyset$.

Answer.

(a) **(2 marks)** The statement is false. For a counterexample take $A = \emptyset$, $B = \emptyset$, and $C = \{\emptyset\}$. Then $A \cap (B \cup C) = \emptyset$ but$(A \cap B) \cup C = \{\emptyset\}$.

(b) **(4 marks)** The statement is true: if $a = m_1^2 + n_1^2$ and $b = m_2^2 + n_2^2$, then

$$ab = (m_1^2 + n_1^2)(m_2^2 + n_2^2) = (m_1 m_2 - n_1 n_2)^2 + (m_1 n_2 + n_1 m_2)^2 .$$

(c) **(2 marks)** The statement is false. For a counterexample take $a = b = 3$: a small argument (done in class!) shows that 3 is not the sum of two integer squares, but $9 = 0^2 + 3^2$ is. [1]

(d) **(2 marks)** The statement is true. For instance $\frac{\varepsilon\sqrt{2}}{2} \in (-\varepsilon, \varepsilon) \setminus \mathbb{Q}$. [2]

[1]NOTE for the marker. I showed in class that 3 is not the sum of two squares. I insist that 1 of 2 marks is for: either producing some kind of argument for this, or stating that it was done in class.

[2]NOTE for the marker. Here it is OK for them to state without proof that $\sqrt{2}$ is not rational.