Question 2.

Let $\alpha = e^{\frac{2\pi i}{5}}$. Find an algebraic equation satisfied by

$$\alpha + \frac{1}{\alpha}$$

Solve this equation and hence compute $\cos \frac{2\pi}{5}$.

Answer.

$$\left(\alpha + \frac{1}{\alpha}\right)^2 = \alpha^2 + 2 + \frac{1}{\alpha^2} = -\alpha - \frac{1}{\alpha} + 1$$

(where we used $0 = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 1 + \left(\alpha + \frac{1}{\alpha}\right) + \left(\alpha^2 + \frac{1}{\alpha^2}\right)$). Hence $\beta = \alpha + \frac{1}{\alpha}$ satisfies the quadratic equation

$$x^2 + x - 1 = 0$$

(5 marks)

The solutions of this equation are

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

(2 marks)

From this we conclude

$$\cos \frac{2\pi}{5} = \Re \alpha = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) = \frac{-1 + \sqrt{5}}{4}$$

(where we took the *positive* root of the above quadratic equation). **(3 marks)**