Name (IN CAPITAL LETTERS!): .... **TID:**

CID: .... **Personal tutor:** ....

Question 3.

Let a, $b \in \mathbb{Z}$. The *least common multiple* of a and b, denoted by $\mathrm{lcm}(a, b)$, is the smallest positive integer that is divisible by both a and b.

Prove, without using unique prime factorization of integers, that if k is divisible by both a and b, then k is divisible by $\mathrm{lcm}(a, b)$.

[*Hint:* prove that, if $a = a' \,\mathrm{hcf}(a, b)$ and $b = b' \,\mathrm{hcf}(a, b)$, then $\mathrm{lcm}(a, b) = \mathrm{hcf}(a, b)a'b'$.]

Answer.

There is no loss of generality in assuming $a, b > 0$ and we do so below.[1] We do this in two steps:

Step 1 5 marks

Write $d = \mathrm{hcf}(a, b)$, $a = da'$, $b = db'$, $m = da'b'$. Then for all k: If k is divisible by a and b, then k is divisible by m.

Indeed, write $k = ap = bq$. We have

$$bq = da'p = db'q, \quad \text{hence} \quad a'p = b'q$$

therefore $a'|b'q$ and by what we did in class, since $\mathrm{hcf}(a', b') = 1$, it follows that $a'|q$. But $k = bq$ and $a'|q$ implies $m = ba'|k$.

Step 2 5 marks

Note that m IS divisible by both a and b, and, since, for $a, b > 0$, $a|b$ implies $a \leq b$, the property in Step 1 implies that $m = \mathrm{lcm}(a, b)$. Then by Step 1 $\mathrm{lcm}(a, b)$ has the stated property.

[1] No marks earned of taken away for this wrinkle.