Name (IN CAPITAL LETTERS!): .... **TID:** ☐

CID: .... **Personal tutor:** ....

Question 1.

(a) Prove that if x, y, z are real numbers such that $x+y+z=0$, then $xy+yz+zx \leq 0$.

(b) For each of the following statements, either prove it or give a couterexample to show that it is false.

(i) The product of two rational numbers is alway rational;

(ii) The product of two irrational numbers is alway irrational;

(iii) The product of two irrational numbers is always rational;

(iv) The product of a non-zero rational and an irrational is always irrational.

Answer.

(a) $\leq (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \geq +(xy+yz+zx)$. 4 marks

(b)

(i) True: it is one of the axioms of $\mathbb{Q}$ that there is a product on it. 1 mark

(ii) False: $x = y = \sqrt{2} \notin \mathbb{Q}$ are both irrational but $xy = (\sqrt{2})^2 = 2$ is not irrational. (If you want $x \neq y$ take $x = \sqrt{2}$ and $y = 2\sqrt{2}$; but note that I didn't ask you for them to be distinct.) 1 mark

(iii) False: take $x = \sqrt{2}$ and $y = \sqrt{3}$. Both x and y are irrational (as shown in class) and their product $xy = \sqrt{6}$ is also irrational (as shown in class). 2 marks

(iv) True. Suppose that x is irrational and $y \neq 0$ is rational. I claim that the product $z = xy$ is irrational. Indeed supposing for a contradiction that z were rational, we would have that

$$x = \frac{xy}{y} = \frac{z}{y} \in \mathbb{Q}$$

(indeed, because of the axioms of $\mathbb{Q}$, $\frac{1}{y} \in \mathbb{Q}$ and then, by the axioms of $\mathbb{Q}$ again, the product $z \times \frac{1}{y} \in \mathbb{Q}$) which contradicts the assumption that x is irrational. 2 marks