Name (IN CAPITAL LETTERS!): .... **TID:** ☐

CID: .... **Personal tutor:** ....

Question 4. For each of the following statements, form its negation and either prove that the statement is true or that its negation is true.

(a) $\exists\, x \in \{y \in \mathbb{N} \mid y \geq 2\}$ such that $\forall\, n \in \mathbb{Z}$, $x \neq n^2 + 2$;

(b) $\forall\, x \in \mathbb{N}$, $\exists\, a, b, c \in \mathbb{Z}$ such that $x = a^2 + b^2 + c^2$;

(c) $\forall\, \varepsilon > 0$, $\forall\, N \in \mathbb{N}$, $\exists\, n \geq N$ such that $n < \frac{1}{\varepsilon}$.

Answer.

(a) Negation: $\forall\, x \in \{y \in \mathbb{N} \mid y \geq 2\}$, $\exists\, n \in \mathbb{Z}$ such that $x = n^2 + 2$. **1** mark

The statement is true: for example $x = 5$ is ≥ 2 and x is not of the form $n^2 + 2$: indeed $3 = 5 - 2$ is not the square of an integer. **2** mark

(b) Negation: $\exists\, x \in \mathbb{N}$ such that $\forall\, a, b, c \in \mathbb{Z}$, $x \neq a^2 + b^2 + c^2$. **1** mark

The negation of the statement is true: indeed $x = 7$ is not the sum of three integer squares. For the proof we may assume (after re-ordering) that $0 \leq a \leq b \leq c$ and then $c^2 \leq 7$ implies $c \leq 2$ so we need to go through all cases $0 \leq a \leq b \leq c \leq 2$:

$$\begin{aligned}
0^2 + 0^2 + 0^2 &= 0 \neq 7 \\
0^2 + 0^2 + 1^2 &= 1 \neq 7 \\
0^2 + 0^2 + 2^2 &= 4 \neq 7 \\
0^2 + 1^2 + 1^2 &= 2 \neq 7 \\
0^2 + 1^2 + 2^2 &= 5 \neq 7 \\
0^2 + 2^2 + 2^2 &= 8 \neq 7 \\
1^2 + 1^2 + 1^2 &= 3 \neq 7 \\
1^2 + 1^2 + 2^2 &= 6 \neq 7 \\
1^2 + 2^2 + 2^2 &= 9 \neq 7 \\
2^2 + 2^2 + 2^2 &= 12 \neq 7
\end{aligned}$$

3 marks

(c) Negation: $\exists\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N$, $n \geq 1/\varepsilon$. **1** mark

The negation of the statement is true: take $\varepsilon = 1$, $N = 1$: then it is true that: If $n \geq 1$, then $n \geq 1$. **2** marks