Name (IN CAPITAL LETTERS!): .... **TID:**

CID: .... **Personal tutor:** ....

Question 4. True or false ? Give a brief (but careful!) proof or counterexample.

(a) $\sqrt{6}+\sqrt{7}<6$.

(b) Every prime number p is the sum of two integer squares (by definition: an integer n is square if and only if f $n=a^2$ where a is integer).

(c) $\forall y \in \{x \in \mathbb{Z} \mid x^2<0\}, 5y^2+5y+1$ is a prime number.

Answer.

(a) True. Indeed, assume for a contradiction that $\sqrt{6}+\sqrt{7} \geq 6$. Since both sides are >0 this implies, squaring both sides, that:

$$6+2\sqrt{42}+7=(\sqrt{6}+\sqrt{7})^2 \geq 6^2=36$$

Rearranging this gives $\sqrt{42} \geq \frac{23}{2}$. This is a contradiction: for example $\sqrt{42}<7<\frac{23}{2}$. **(4 marks)**

(b) False. A counterexample to the statement is $p=3$. We need to see that: $\forall\ a,b \in \mathbb{N},\ a^2+b^2 \neq 3$. To verify this we only need to check $a,b\ : a^2,b^2 \leq 3$, that is, $a,b \in \{0,1\}$, which implies $a^2+b^2=0,1,2$. **(3 marks)**

(c) True. The set $\{x \in \mathbb{Z} \mid x^2<0\}$ is the empty set; therefore it is a subset of every set, including the set $\{y \in \mathbb{Z} \mid 5y^2+5y+1 \text{ is prime}\}$. **(3 marks)**