M1F Foundations of Analysis, Problem Sheet 9, solutions.

1.

(i) If $a \in \mathbf{R}$ then $a - a = 0 \in G$, so $a \sim a$. But $a \in \mathbf{R}$ was arbitrary, so $\sim$ is reflexive.

(ii) Say $a, b \in \mathbf{R}$ and $a \sim b$. Then $g := b - a \in G$ by definition. So $-g \in G$, so $a - b \in G$, so $b \sim a$. But a, b were arbitrary, so $\sim$ is symmetric. (iii) Say $a, b, c \in \mathbf{R}$ and $a \sim b$ and $b \sim c$. Then $g := b - a \in G$ and $h := c - b \in G$, hence $g + h \in G$. But $g + h = c - a$, so $a \sim c$. But a, b, c were arbitrary, so $\sim$ is transitive.

(iv) If $\sim$ is reflexive then $0 \sim 0$ so $0 - 0 \in G$ hence $0 \in G$. If $\sim$ is symmetric then for $g \in G$ we have $0 \sim g$, so $g \sim 0$, so $-g \in G$. Finally if $\sim$ is transitive then for $g, h \in G$ we have $0 \sim g$ and $g \sim g + h$, so $0 \sim g + h$ hence $g + h \in G$.

2* Some of this stuff is only a mild extension of what I did in lectures, but rather than referring you to the proofs in lectures I'm just going to do everything in full again.

(i) Note that $a \sim b$ if and only if $a \equiv b \bmod 2$ does define an equivalence relation (by 6.12). If $a \in \mathbf{Z}$ then $\mathrm{cl}(a)$ means all the $b \in \mathbf{Z}$ such that $a \equiv b \bmod 2$, which is the integers b such that $b - a$ is even (by definition). If a is even then $b - a$ is even iff b is even, so $\mathrm{cl}(a)$ is all the even integers. Conversely if a is odd then $b - a$ is even iff b is odd, so $\mathrm{cl}(a)$ is all the odd integers. So every equivalence class is either the set of all even integers, or the set of all odd integers, and in particular there are only two choices.

(ii) Sure this is possible. For example $\mathrm{cl}(1) = \mathrm{cl}(3)$ (both of these sets are all the odd integers).

(iii) This is not possible. The only possibilities are that X is either the set of all odd integers, or the set of all even integers; the same is true for Y. So if $X \neq Y$ then one had better be all the even integers and the other all the odd integers, which forces $X \cap Y = \emptyset$.

(iv) Say $\mathrm{cl}(a) = \mathrm{cl}(b)$. Recall that $\mathrm{cl}(b) = \{c \in S : b \sim c\}$. We know $\sim$ is an equivalence relation, so $b \sim b$, so $b \in \mathrm{cl}(b)$. If we're assuming $\mathrm{cl}(a) = \mathrm{cl}(b)$ then this means $b \in \mathrm{cl}(a)$, so by definition we have $a \sim b$.

The other way I did in lectures; if $a \sim b$ then for any $c \in \mathrm{cl}(b)$ we have $b \sim c$ (by definition) so $a \sim c$ (by transitivity), so $c \in \mathrm{cl}(a)$. This shows $\mathrm{cl}(b) \subseteq \mathrm{cl}(a)$. I'll now explicitly do the other way: say $x \in \mathrm{cl}(a)$; then $a \sim x$, so $x \sim a$ (symmetry) so $x \sim b$ (transitivity) so $b \sim x$ (symmetry) so $x \in \mathrm{cl}(b)$. Hence $\mathrm{cl}(a) \subseteq \mathrm{cl}(b)$ and so $\mathrm{cl}(a) = \mathrm{cl}(b)$.

(v) If there exists some $s \in \mathrm{cl}(a) \cap \mathrm{cl}(b)$ then $a \sim s$ and $b \sim s$; by symmetry $s \sim b$ and by transitivity $a \sim b$, so (a) implies (b).

We just proved (b) implies (c) in (iv).

For (c) implies (a) we note that $a \in \mathrm{cl}(a)$ by reflexivity, and so $a \in \mathrm{cl}(b)$ meaning $\mathrm{cl}(a) \cap \mathrm{cl}(b) \neq \emptyset$, so (c) implies (a).

Thus (a), (b) and (c) are all equivalent.

3.

(i) To prove that it *is* possible to show that reflexive and transitive implies symmetric, we just have to write down a proof that if $\sim$ is any binary relation on any set S and $\sim$ is reflexive and transitive, then $\sim$ is symmetric. To prove that it is *not* possible, we need to write down a counterexample, which in this context would mean an example of a set S and a binary relation $\sim$ which is reflexive and is transitive but is not symmetric.

(ii) We know $a \leq a$ for all a, so $\sim$ is reflexive. We also know that $a \leq b$ and $b \leq c$ implies $a \leq c$, so $\sim$ is transitive. But $\sim$ is not symmetric because $3 \sim 4$ but $4 \not\sim 3$.

(iii) If $x \in \mathbf{R}$ then $|x - x| = 0 \leq 1$, so $x \sim x$. Hence $\sim$ is reflexive. Next, if $|x - y| \leq 1$ then $|y - x| = |-(x - y)| = |x - y| \leq 1$, so $\sim$ is symmetric. However $1 \sim 2$ and $2 \sim 3$ but $1 \not\sim 3$, so $\sim$ is not transitive.

(iv) $0 \sim 0$ is false, so $\sim$ is not reflexive. However $\sim$ is symmetric and transitive, because whatever x and y are, $(x \sim y)$ is always false, so $(x \sim y) \implies (y \sim x)$ is true (because false implies anything). Similarly transitivity is true (because false implies anything).

4. The mistake is that you might not be able to choose t with $s \sim t$, because perhaps there are no $t \in S$ at all (including $t = s$) satisfying $s \sim t$.

5.

(i) This is straightforward: $f(a) = f(a)$ so $a \sim a$, if $a \sim b$ then $f(a) = f(b)$ so $f(b) = f(a)$ so $b \sim a$, and if $a \sim b$ and $b \sim c$ then $f(a) = f(b) = f(c)$ so $f(a) = f(c)$ so $a \sim c$.

(ii) $\mathrm{cl}(x) = \{s \in X : x \sim s\} = \{s \in X : f(x) = f(s)\} = \{s \in X : f(s) = y\} = f^{-1}(y)$.

(iii) Say W is an equivalence class. By definition this means $W = \mathrm{cl}(x)$ for some $x \in X$. Now choose $w \in W$. Then $x \sim w$ so $W = \mathrm{cl}(w)$ by a previous question. I want to define $g(W) = f(w)$. So now let's see what happens if we choose $w' \in W$. Then $w \sim w'$ because $W = \mathrm{cl}(w)$, and hence $f(w) = f(w')$ by definition of $\sim$. In particular this means that our definition of $g(W)$ was indeed independent of the choice of element of W we used to define $g(W)$, so $g(W)$ is indeed well-defined.

(iv) Say W_1 and W_2 are two equivalence classes. Choose $w_1 \in W_1$ and $w_2 \in W_2$. Then W_1 has non-trivial intersection with $\mathrm{cl}(w_1)$ (they both contain w_1) so $\mathrm{cl}(w_1) = W_1$ and similarly $\mathrm{cl}(w_2) = W_2$. By definition, $g(W_1) = f(w_1)$ and $g(W_2) = f(w_2)$. Now let's say $g(W_1) = g(W_2)$. Then $f(w_1) = f(w_2)$, so $w_1 \sim w_2$, so $\mathrm{cl}(w_1) = \mathrm{cl}(w_2)$. Hence $W_1 = W_2$. But W_1 and W_2 were arbitrary, so g is injective.

(v) Say $W \in Z$. Then by definition W is an equivalence class, so by definition $W = \mathrm{cl}(x)$ for some $x \in X$. Hence $h(x) = \mathrm{cl}(x) = W$. But $W \in Z$ was arbitrary, so h is surjective.

(vi) To prove that $f = g \circ h$ we need to check that for all $x \in X$ we have $f(x) = g(h(x))$. So set $W = h(x) = \mathrm{cl}(x)$. Now to define $g(W)$ we need to choose some element of W but we know for sure (by reflexivity) that $x \in \mathrm{cl}(x)$ so let's choose x, and then $g(W) = f(x)$. Hence $g(h(x)) = f(x)$. But $x \in X$ was arbitrary, so $g \circ h = f$.

(vii) Say $j \in J$. Then by definition of the image of f we must have $j = f(x)$ for some $x \in X$. Define $i(j) = f^{-1}(j)$; by part (ii) this is $\mathrm{cl}(x)$. In particular i is a well-defined map.

The reason it is injective is that if $j_1 \neq j_2 \in J$ then we can choose $x_1, x_2 \in X$ such that $f(x_1) = j_1$ and $f(x_2) = j_2$; now $x_1 \not\sim x_2$ (as $j_1 \neq j_2$) so $\mathrm{cl}(x_1) \neq \mathrm{cl}(x_2)$, meaning that indeed i is injective.

The reason i is surjective, is that if W is an equivalence class and $w \in W$ then $f(w) = j \in J$ and $i(j) = f^{-1}(j)$ is an equivalence class containing w so must be $\mathrm{cl}(w) = W$.

Hence i is bijective.

(viii) If $x \in X$ and $f(x) = j$ then $i(\tilde{h}(x)) = i(f(x)) = i(j) = f^{-1}(j) = \{x' \in X : f(x') = f(x)\} = \mathrm{cl}(x) = h(x)$. Because $x \in X$ was arbitrary we have proved $i \circ \tilde{h} = h$.

Finally, if $j \in J$ then by definition $j = f(x)$ for some $x \in X$. If $i(x) = W$ then $W = f^{-1}(j)$ which contains x, so $g(i(j)) = g(W) = f(x) = j = \tilde{g}(j)$, and because $j \in J$ was arbitrary we have proved $g \circ i = \tilde{g}$.

6.

(i) $a + b = b + a$ so $\sim$ is reflexive. If $(a, b) \sim (c, d)$ then $a + d = b + c$ so $c + b = d + a$ so $(c, d) \sim (a, b)$, so $\sim$ is symmetric. Finally if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ then $a + d = b + c$ and $c + f = d + e$ so $a + f = a + d + c + f - (c + d) = b + c + d + e - (c + d) = b + e$, and hence $\sim$ is transitive. Thus $\sim$ is an equivalence relation.

(ii) f is clearly surjective, because if $n \geq 0$ then $f((n, 0)) = n$ and if $n = -d < 0$ then $f((0, d)) = n$. Furthermore, $(a, b) \sim (c, d)$ iff $a + d = b + c$ iff $a + d - (b + d) = b + c - (b + d)$ iff $a - b = c - d$ iff $f((a, b)) = f((c, d))$.

(iii) Let Z be the set of equivalence classes for this equivalence relation. Let's define $i : Z \to \mathbf{Z}$ thus: if $W \in Z$ then choose $(a, b) \in W$ and define $i(W) = f((a, b)) = a - b$. We need to check that this is a well-defined function, because we made a choice here. But the previous part of this question did this: if we had instead chosen $(c, d) \in W$ then by definition $(a, b) \sim (c, d)$ so $a - b = c - d$. In particular i is well-defined. It is also surjective because if $n \in \mathbf{Z}$ then we showed in part (ii) that $n = f((a, b))$ for some (a, b) and hence $n = i(W)$ if $W = \mathrm{cl}((a, b))$. Finally, it is injective because if $i(W_1) = i(W_2)$ then, choosing $(a_1, b_1) \in W_1$ and $(a_2, b_2) \in W_2$ we have $f((a_1, b_1)) = f((a_2, b_2))$ so $(a_1, b_1) \sim (a_2, b_2)$ so $\mathrm{cl}((a_1, b_1)) = \mathrm{cl}((a_2, b_2))$ so $W_1 = W_2$. Hence it is bijective.

(iv) One checks that $\sim$ is an equivalence relation, that if $f : S \to \mathbf{Q}$ is defined by $f((a,b)) = a/b$ then f is surjective, and $f((a,b)) = f((c,d))$ if and only if $(a,b) \sim (c,d)$; one then checks that f induces a bijection between the equivalence classes and $\mathbf{Q}$. So one could define $\mathbf{Q}$ as the equivalence classes, if one had not got a definition of $\mathbf{Q}$ already.

7. If $x \in \mathbf{Z}$ then $x \sim x+8 \sim x+16$ (using $n = x$ and $n = x+8$) and similarly $x+1 \sim x+6 \sim x+11 \sim x+16$. By symmetry $x+16 \sim x+1$ and by transitivity $x \sim x+1$ for all $x \in \mathbf{Z}$. Now by induction we can prove that if y is fixed and $z = y+n$ for some integer $n \geq 0$ then $y \sim z$. The base case is reflexivity, and the inductive step follows from the fact that $x \sim x+1$. As a consequence we deduce that if $y \leq z$ then $y \sim z$. By symmetry we deduce that if $y \leq z$ then $y \sim z$, and hence $y \sim z$ for all $y, z \in \mathbf{Z}$.