

**M1F Foundations of Analysis, Problem Sheet 2.**

1. Recall that to check that two sets  $A$  and  $B$  are equal, one has to do two things: first prove  $A \subseteq B$  and then prove  $B \subseteq A$ .

(a)  $\bigcup_{n=0}^{\infty} [n, n+1)$  equals  $[0, \infty)$ . Why is this? Because  $n \geq 0$  in the union, we have  $0 \leq n < n+1 < \infty$ , so certainly the union is contained within  $[0, \infty)$ . Conversely if  $r \in (0, \infty)$  then there is some integer  $n$  such that  $n \leq r < n+1$  (we'll prove this in the course; alternatively you might want to argue that it's "obvious" and whether it is or not depends on your viewpoint of what mathematics is). This integer  $n$  must be at least zero, as if  $n < 0$  then  $n \leq -1$ , so  $n+1 \leq 0$ , which implies  $r < n+1 \leq 0$ , a contradiction. Hence  $r \in [n, n+1)$  and  $n \geq 0$ , so this is in the union on the left hand side.

(b) This union is  $(0, 1]$ . For if  $n \geq 1$  then  $1/n > 0$  and hence  $[1/n, 1] \subseteq (0, 1]$ , so we can deduce that the union is contained within  $(0, 1]$ . Conversely, if  $r > 0$  then we showed in lectures that there's some positive integer  $n$  with  $0 < 1/n < r$  (or maybe this is "obvious"), and hence  $r \in [1/n, 1]$ .

(c) This union is all of  $\mathbf{R}$ . It's clearly contained in  $\mathbf{R}$ , and conversely if  $r$  is any real number and we choose an integer  $N > 0$  with  $N > r$ , and an integer  $M > 0$  with  $M > -r$ , and let  $n$  be the maximum of  $N$  and  $M$ , we have  $r < N \leq n$  and  $-r < M \leq n$  so  $-n < r$ , and we conclude  $r \in (-n, n)$ .

(d) The intersection is just  $(-1, 1)$ . For if  $n \geq 1$  then  $-n \leq -1 < 1 \leq n$  and hence  $(-1, 1) \subseteq (-n, n)$ , meaning that  $(-1, 1)$  is contained in the intersection; conversely the intersection is contained in each of the sets in the intersection and in particular within  $(-1, 1)$ .

2) Informally, we are going to argue that there can be no largest element, because if  $s$  is in  $(0, 1)$  then the average of  $s$  and 1 will be a bit larger. Let me write this down more formally though.

We prove the result by contradiction. Let's assume for a contradiction that  $s$  is a largest element of  $(0, 1)$ . Then let's consider  $t := \frac{s+1}{2}$ . Because  $s \in (0, 1)$  we have  $s < 1$ , and hence  $s+1 < 2$  so  $t = \frac{s+1}{2} < 1$ . Because  $s > 0$  we have  $s+1 > 0$  and hence  $t = \frac{s+1}{2} > 0$ . We deduce that  $t \in (0, 1)$ . Now if  $s$  were a largest element of  $(0, 1)$  then we must have  $t \leq s$ , but I claim that in fact  $t > s$ . For  $t - s = \frac{s+1}{2} - \frac{2s}{2} = \frac{1-s}{2} > 0$  because  $s < 1$  hence  $1-s > 0$ .

3) By contradiction. Let's say 3 divides  $n^2$  but it doesn't divide  $n$ . Then the remainder when we divide  $n$  by 3 must be 1 or 2, in other words  $n = 3m+1$  or  $3m+2$ .

In the first case  $n^2 = (3m+1)^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1$  is not a multiple of 3.

In the second case  $n^2 = (3m+2)^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1$  is also not a multiple of 3.

So in either case we have our contradiction, meaning that if 3 divides  $n^2$  then 3 must divide  $n$ .

4) (a) This is false. For example if  $a = \sqrt{2}$  and  $b = -\sqrt{2}$  then both  $a$  and  $b$  are irrational, but their sum is zero, which is rational.

(b) This is also false. For example if  $a = \sqrt{2}$  and  $b = 0$  then  $ab = 0$  is rational.

5) (a) This is true. We need to show that if  $x \in \mathbf{R}$  is arbitrary, then there exists some  $y \in \mathbf{R}$  such that  $x + y = 2$ , and this is easy: we can just let  $y = 2 - x$ .

(b) This is not true. The claim is that there is some magical number  $y \in \mathbf{R}$  which has the property that whatever real number  $x$  we choose, we will have  $x + y = 2$ . But this cannot be true. Let's prove it by contradiction. Let's assume for a contradiction that such a number  $y$  really did exist, and now let's try some values of  $x$ . For example let's choose  $x = 0$ ; then we have  $y + 0 = 2$  and hence  $y = 2$ . But now let's choose  $x = 1$ ; then we must have  $2 + 1 = y + 1 = 2$ , and hence  $3 = 2$ , a contradiction. So no such  $y$  can exist.

(6) Note that  $\sqrt{2}$ ,  $\sqrt{6}$  and  $\sqrt{15}$  are all positive. Let's prove  $\sqrt{2} + \sqrt{6} < \sqrt{15}$  by contradiction. So let's assume

$$\sqrt{2} + \sqrt{6} \geq \sqrt{15}$$

(NB lose a mark for  $>$ ; the opposite of  $<$  is  $\geq$ ).

Both sides are positive so we can square both sides and deduce

$$(\sqrt{2} + \sqrt{6}) \geq 15.$$

Now expand out the bracket and tidy up, to get

$$2\sqrt{12} \geq 15 - 8 = 7.$$

Again both sides are positive so we can square both sides and conclude

$$48 \geq 49$$

and this is a contradiction.

Hence  $\sqrt{2} + \sqrt{6} < \sqrt{15}$ .

**IMPORTANT NOTE.** If you wrote something like this:

$$\begin{aligned}\sqrt{2} + \sqrt{6} &< \sqrt{15} \\ \Rightarrow 2 + 6 + 2\sqrt{12} &< 15 \\ \Rightarrow 2\sqrt{12} &< 7 \\ \Rightarrow 48 &< 49\end{aligned}$$

then you get no marks at all. This is because if  $P$  is the statement that  $\sqrt{2} + \sqrt{6} < \sqrt{15}$  then the argument just above shows that  $P$  implies  $48 < 49$ , so  $P$  implies something true. What can we deduce about  $P$  from this? Nothing! Because true implies true, and false implies true.

If however you wrote

$$\begin{aligned}\sqrt{2} + \sqrt{6} &< \sqrt{15} \\ \Leftarrow 2 + 6 + 2\sqrt{12} &< 15 \\ \Leftarrow 2\sqrt{12} &< 7 \\ \Leftarrow 48 &< 49\end{aligned}$$

then this would logically be fine, although arguably it would also be upside-down, and also strictly speaking it doesn't follow from what we proved in the course about inequalities because we only proved  $0 < a < b$  implies  $0 < a^2 < b^2$  rather than the other way around. Can you see how to prove that  $0 < a^2 < b^2$  and  $a, b > 0$  implies  $a < b$ ?

7) (a) Proof by contradiction. If  $\sqrt{2} + \sqrt{3/2}$  were rational, then its square would be too. But its square is  $2 + 3/2 + 2\sqrt{3}$ , and if this were rational then  $2\sqrt{3}$  and hence  $\sqrt{3}$  would be too, contradicting Q3.

(b) This must be irrational because if it were rational then adding  $-1$  would leave it rational, but adding 1 gives part (a) which is irrational.

(c) This is rational because it's zero :P