## M1F Foundations of Analysis, Problem Sheet 8.

**1.** Let $a$ and $b$ be coprime positive integers (recall that *coprime* here means $\gcd(a,b)=1$). I open a fast food restaurant which sells chicken nuggets in two sizes – you can either buy a box with $a$ nuggets in, or a box with $b$ nuggets in. Prove that there is some integer $N$ with the property that for all integers $m \geq N$, it is possible to buy exactly $m$ nuggets.

**$2^*$.** True or false?

(i) If $a$ and $b$ are positive integers, and there exist integers $\lambda$ and $\mu$ such that $\lambda a+\mu b=1$, then $\gcd(a,b)=1$.

(ii) If $a$ and $b$ are positive integers, and there exist integers $\lambda$ and $\mu$ such that $\lambda a+\mu b=7$, then $\gcd(a,b)=7$.

**3.** (the "rule of $n$" for various integers $n$). You know that a positive integer is even if and only if its last digit is even. This is because if $N$ is a positive integer with last digit $d$, then $N-d$ ends in zero so is a multiple of 10 and hence a multiple of 2, so $N$ is even if and only if $d$ is even. Let's find some more rules for working out if a number is divisible by $n$ for various other small values of $n$.

(i) Prove (either by induction or directly) that $10^e \equiv 1 \bmod 9$ for all $e$. Deduce that if $N$ is any positive integer and the sum of the digits of $N$ is $s$, then $N-s$ is a multiple of 9. Deduce that $N$ is a multiple of 9 if and only if $s$ is. Is 123456789 a multiple of 9?

(ii) Deduce from (i) that $N$ is a multiple of 3 if and only if $s$ is.

(iii) Prove (either by induction or directly) that if $r \in \mathbf{Z}_{\geq 1}$ and $e \geq r$ then $10^e \equiv 0 \bmod 2^r$. Deduce that if $N$ is any positive integer and $s$ is the number which is the last $r$ digits of $N$, then $N-s$ is a multiple of $2^r$. Deduce that $N$ is a multiple of $2^r$ if and only if the last $r$ digits of $N$ is a multiple of $2^r$. Is 3847534875634765124 a multiple of 4? Is 9995763848388 a multiple of 8?

(iv) Prove (either by induction or directly) that if $e \in \mathbf{Z}_{\geq 1}$ then $10^e \equiv (-1)^e \bmod 11$. Deduce that if $N$ is any positive integer and $s$ is the number you get by doing (last digit of $N$) minus (second-last digit of $N$) add (third-last digit of $N$) minus…, then $N-s$ is a multiple of 11. What is the remainder when 1234567 is divided by 11?

**4.** (i) Say $a$ and $b$ are coprime positive integers, and $N$ is any integer which is a multiple of $a$ and of $b$. Prove that $N$ is a multiple of $ab$. Hint: we know that $\lambda a+\mu b=1$ for some $\lambda,\mu \in \mathbf{Z}$; now write $N = N \times (\lambda a+\mu b)$.

(ii) By applying (i) twice, deduce that if $p$, $q$ and $r$ are three distinct primes, then two integers $x$ and $y$ are congruent modulo $pqr$ if and only if they are congruent mod $p$, mod $q$ and mod $r$.

(iii) (tough) Consider the set of positive integers $\{2^7-2, 3^7-3, 4^7-4, \ldots, 1000^7-1000\}$. What is the greatest common divisor of all the elements of this set? Feel free to use a calculator to get the hang of this; feel free to use Fermat's Little Theorem and the previous part to nail it.

(iv) (tougher) $561 = 3 \times 11 \times 17$. Prove that if $n \in \mathbf{Z}$ then $n^{561} \equiv n \bmod 561$. Hence the converse to Fermat's Little Theorem is false.

**6.** For each of the following binary relations on a set $S$, figure out whether or not the relation is reflexive. Then figure out whether or not it is symmetric. Finally figure out whether or not the relation is transitive.

(i) $S=\mathbf{R}$, $a \sim b$ if and only if $a \leq b$.

(ii) $S=\mathbf{Z}$, $a \sim b$ if and only if $a-b$ is the square of an integer.

(iii) $S=\mathbf{R}$, $a \sim b$ if and only if $a=b^2$.

(iv) $S=\mathbf{Z}$, $a \sim b$ if and only if $a+b=0$.

(v) $S=\mathbf{R}$, $a \sim b$ if and only if $a-b$ is an integer.

(vi) $S=\{1,2,3,4\}$, $a \sim b$ if and only if $a=1$ and $b=3$.

(vii) $S$ is the empty set (and $\sim$ is the only possible binary relation on that set, the empty binary relation).