KMB, 12/12/16

M1F Foundations of Analysis, Problem Sheet 10.

1*. (hard) If you have never spoken to me before, then ask me a question about the course in the problems class on Thursday, or in my office hour just before the problems class.

2. For each of the following functions, decide whether or not they are injective, surjective, bijective. Proofs required!

(i) $f : \mathbf{R} \to \mathbf{R}$, $f(x) = 1/x$ if $x \neq 0$ and $f(0) = 0$.

(ii) $f : \mathbf{Z} \to \mathbf{Z}$, $f(n) = 2n + 1$.

(iii) $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^3$. [you can assume that every positive real has a unique positive real cube root, even though you haven't really seen a formal proof of this yet.]

(iv) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$ if the Riemann hypothesis is true, and $f(x) = -x$ if not. [NB the Riemann Hypothesis is a hard unsolved problem in mathematics; nobody currently knows if it is true or false.]

(v) (hard) $f : \mathbf{Z} \to \mathbf{Z}$, $f(n) = n^3 - 2n^2 + 2n - 1$.

3. For each of the following "functions", explain why I just lost a mark.

(i) $f : \mathbf{R} \to \mathbf{R}$, $f(x) = 1/x$.

(ii) $f : \mathbf{R} \to \mathbf{R}$, $f(x) = \sqrt{x}$.

(iii) $f : \mathbf{Z} \to \mathbf{Z}$, $f(n) = (n + 1)^2/2$.

(iv) $f : \mathbf{R} \to \mathbf{R}$, $f(x)$ is a solution to $y^3 - y = x$.

(v) $f : \mathbf{R}\backslash\{1\} \to \mathbf{R}$, $f(x) = 1 + x + x^2 + x^3 + \cdots$.

3. (One-sided inverses.)

(i) Say $f : X \to Y$ is a function and there exists a function $g : Y \to X$ such that $f \circ g$ is the identity function $Y \to Y$. Prove that f is surjective.

(ii) Say $f : X \to Y$ is a function and there exists a function $g : Y \to X$ such that $g \circ f$ is the identity function $X \to X$. Prove that f is injective.

4*. This relatively straightforward (if you've understood the idea correctly) question makes sure that you understand the composition of functions, and how it differs from other things like multiplication of functions.

Say f and g are functions from $\mathbf{R}$ to $\mathbf{R}$ defined by $f(x) = x^2 + 3$ and $g(x) = 2x$. Write down explicit formulae for the following functions:

(i) $f \circ g$

(ii) $g \circ f$

(iii) $x \mapsto f(x)g(x)$

(iv) $x \mapsto f(x) + g(x)$

(v) $x \mapsto f(g(x))$.

5. Say A and B are finite sets, with sizes a and b respectively. Prove that the set B^A of functions $A \to B$ has size b^a. What about the case $a = b = 0$?

6. (converse of 3 with added bits.)

(i) Say $f : X \to Y$ is surjective. Prove that there exists a function $g : Y \to X$ such that $f \circ g$ is the identity function. Bonus point: in general, is g "natural"?

(ii) Say $f : X \to Y$ is injective. Does there always exist a function $g : Y \to X$ such that $g \circ f$ is the identity function?

(iii) Say $f : X \to Y$ is any function. Does there always exist an injection $g : X \to Z$ and a surjection $h : Z \to Y$ such that $f = h \circ g$?

7. Prove that $\circ$ is associative. In other words, prove that if $h : A \to B$ and $g : B \to C$ and $f : C \to D$ then $(f \circ g) \circ h = f \circ (g \circ h)$ (NB these are both functions $A \to D$). This is a great example of a question that is dead easy once you actually figure out what it's asking.

8. (i) Prove that if X is a countably infinite set, and $Y \subseteq X$ is an infinite subset, then Y is countably infinite. (ii) Are there countably many irrational numbers? Countably many complex numbers? What about $\mathbf{Q}(i) = \{a + bi \,:\, a, b \in \mathbf{Q}\}$?