M1F Foundations of Analysis — Problem Sheet 9

1. Show that for complex numbers x, y we have $\overline{x+y} = \overline{x} + \overline{y}$ and $\overline{xy} = \overline{x}.\overline{y}$.

   Just express in Cartesian form, $x = a + ib$, etc, and compute both sides.

2. * For which $z, w \in \mathbb{C}$ do we have $\overline{z + iw} = z - iw$?

   By Q1, $\overline{z+iw} = \overline{z} - i\overline{w}$ so this equals $z - iw$ if and only if

   $$\overline{z} - z = i(\overline{w} - w).$$

   The left hand side is entirely imaginary (it's $-2i\text{Im}(z)$) while the right hand side is entirely real (it's $2\text{Im}(w)$). So to be equal they must both be zero. Therefore z, w solve the equation if and only

   $$\overline{z} = z \qquad \text{and} \qquad \overline{w} = w,$$

   if and only if both z and w are real.

3. (a) What is $\sqrt{i}$?

   (b) Find all the 5th roots of $16(1 + \sqrt{3}i)$.

   (c) Write $(1+i)(1-\sqrt{3}i)$ in the form $x + iy$ and in polar form. Deduce the value of $\sin(\pi/12)$.

   $i = e^{i\pi/2} = e^{5i\pi/2}$ **so the square roots of i are $e^{i\pi/4}$ and $e^{5i\pi/4}$. I.e. they are $\pm\frac{1}{\sqrt{2}}(1+i)$.**

   $16(1+\sqrt{3}i) = 32(1/2 + \sqrt{3}/2i) = 2^5 e^{i\pi/3}$ **with 5th roots $2e^{i\pi/15 + 2ki\pi/5}$, $k = 0, 1, 2, 3, 4$.**

   $(1+i)(1-\sqrt{3}i) = 2\sqrt{2}\,\frac{1+i}{\sqrt{2}}\,\frac{1-\sqrt{3}i}{2} = 2^{3/2}e^{i\pi 4}e^{-i\pi/3} = 2^{3/2}e^{-i\pi/12}$ **in polar form.**

   Multiplying out, it is also $(1+\sqrt{3}) + i(1-\sqrt{3})$ in Cartesian form.

   Comparing imaginary parts, we see that $\sin(\pi/12) = -2^{-3/2}(1-\sqrt{3})$.

4. Given a complex number $u \neq 1$, let $z = \frac{1+u}{1-u}$.

   Show that $z = -\overline{z} \iff u\overline{u} = 1$.

   As u runs round the unit circle, what does z do ?

   $z = -\overline{z}$ **iff** $\frac{1+u}{1-u} = \frac{1+\overline{u}}{\overline{u}-1}$ **iff** $\overline{u} + u\overline{u} - 1 - u = 1 - u + \overline{u} - u\overline{u}$ **iff** $2u\overline{u} = 2$.

   u lies in the unit circle iff $|u|^2 = 1$ iff $u\overline{u} = 1$ iff $z = -\overline{z}$ iff z is imaginary. So z runs up the imaginary axis.

5. State the fundamental theorem of algebra and factorise

   $$p(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n.$$

   Hence write a_1 in terms of the roots of p. Hence give another proof that the roots of $z^n = 1$ add up to zero when $n \geq 2$.

   There exist (not necessarily distinct) $\lambda_i \in \mathbb{C},\ i = 1, \ldots, n$ such that $p(z) = (z-\lambda_1)(z-\lambda_2)\ldots(z-\lambda_n)$.

   Multiplying out gives $p(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n = z^n - (\lambda_1 + \lambda_2 + \ldots + \lambda_n)z^{n-1} + \ldots$, so comparing coefficients of z^{n-1} gives

   $$a_1 = -(\lambda_1 + \lambda_2 + \ldots + \lambda_n).$$

   Applied to the polynomial $p(z) = z^n - 1$ with $a_1 = 0$ (if $n \geq 2$) shows that the roots of unity sum to 0.

6. Find the solutions of $1 + z + z^2 + \ldots + z^{n-1} = 0$.

$(1-z)(1+z+z^2+\ldots+z^{n-1}) = 1 - z^n$ has roots $z = e^{2k\pi i/n}$, $k = 0, 1, \ldots, n-1$. The first factor $(z-1)$ gives the root $z = 1$, i.e. $k = 0$. So the rest are the roots of $1 + z + z^2 + \ldots + z^{n-1} = 0$, i.e.

$$z = e^{2k\pi i/n}, \quad k = 1, 2, \ldots, n-1.$$

7. There was a quadratic equation on my desk when I went to lunch, and now I can't find it anywhere. I can remember its roots – λ and μ – but not the equation. I try to cheer myself up by solving another one,

$$Ax^2 + Bx + C = 0$$

and I notice that its roots are $-(\lambda + \mu)^2$ and $\lambda^2 + \mu^2$. What was the equation that I lost ?

Since $B/A = (\lambda+\mu)^2 - (\lambda^2+\mu^2)$ and $C/A = -(\lambda+\mu)^2(\lambda^2+\mu^2)$ we find that

$$B/A = 2\lambda\mu$$

and $C/A = -s^2(s^2 - B/A)$, where $s := \lambda + \mu$. Thus

$$s^4 - (B/A)s^2 + C/A = 0 \quad \Longrightarrow \quad s = \pm\sqrt{\frac{B \pm \sqrt{B^2 - 4AC}}{2A}}.$$

Therefore we cannot know the original equaton exactly, but it is one of the 4 equations

$$x^2 \mp \sqrt{\frac{B \pm \sqrt{B^2 - 4AC}}{2A}}\, x + \frac{B}{2A} = 0.$$

*You should prepare starred questions * to discuss with your personal tutor.*