# M1F Foundations of Analysis — Problem Sheet 8

1. Consider functions $f, g\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ g(x) = (x-1)^2$.

   (a) If $g(x) = x^2$ show that there is no solution for $f$. (Before you think of this you may go back to Q4 of sheet 6b.)

   (b) If $g(x) = x^2$, you have just shown that there are no solutions for $f$. So what is wrong with $f(x) := \begin{cases} x - 2\sqrt{x} + 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ (where $\sqrt{x}$ denotes the *positive* square root) ?

   (c) If $f(x) = x^2$, then what are the possibilities for $g$ ? How many are there ?

   (d) If $f(x) = x^2 + 2x - 1$, then what are the possibilities for $g$? (Before you think about this go back to Q2 of sheet 6b.) How many are there ?

**(a) Since $g(x) = x^2$ then $g(x) = g(-x)$ so in particular $f(g(1)) = f(g(-1))$. But we are told that $f \circ g(1) = 0 \neq 4 = f \circ g(-1)$.**

**(b) If we set $f(x) := \begin{cases} x - 2\sqrt{x} + 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ then we find that $f(g(x)) = f(x^2) = x^2 - 2|x| + 1 = (|x|-1)^2$, which gives the wrong answer for $x < 0$.**

**(c) The possibilities are those $g$ such that $(g(x))^2 = (x-1)^2$ for all $x \in \mathbb{R}$. So we know that for each $x$, $g(x)$ is either $x-1$ or $1-x$. Therefore there are an *INFINITE* number of such $g$, with two different choices of $g(x)$ for every $x \neq 1$ (and $g(1) = 0$).**

**(d) The possibilities are those $g$ such that $(g(x))^2 + 2g(x) - 1 - (x-1)^2 = 0$ for all $x \in \mathbb{R}$. Therefore $g(x) = -1 \pm \sqrt{2 + (x-1)^2}$, which is two different real numbers (since $2 + (x-1)^2 > 0$). Therefore there are an *INFINITE* number of such $g$, with two different choices of $g(x)$ for every $x \in \mathbb{R}$.**

2. $*$ Fix $S \subset \mathbb{R}$ with an upper bound, and suppose that $S \neq \emptyset$ and $S \neq \mathbb{R}$. Give proofs or counterexamples to the following statements.

   (a) If $S \subset \mathbb{Q}$ then $\sup S \in \mathbb{Q}$.

   (b) If $S \subset \mathbb{R}\backslash\mathbb{Q}$ then $\sup S \in \mathbb{R}\backslash\mathbb{Q}$.

   (c) If $S \subset \mathbb{Z}$ then $\sup S \in \mathbb{Z}$.

   (d) There exists a $\max S$ if and only if $\sup S \in S$.

   (e) $\sup S = \inf(\mathbb{R}\backslash S)$.

   (f) $\sup S = \inf(\mathbb{R}\backslash S)$ if and only if $S$ is an interval of the form $(-\infty, a)$ or $(-\infty, a]$.

**(a) False, eg $S = \{x \in \mathbb{Q} \colon x < \sqrt{2}\}$ with $\sup S = \sqrt{2}$.**

**(b) False, eg $S = \{x \in \mathbb{R}\backslash\mathbb{Q} \colon x < 0\}$ with $\sup S = 0$.**

**(c) True. Pick $s_0 \in S$ and an integer $N$ larger than a given upper bound for $S$. Then $[s_0, N] \cap \mathbb{Z}$ is a finite set, so $[s_0, N] \cap S$ is also finite and nonempty set of integers so has a maximum $m \in \mathbb{Z}$. By (d) this is also $\sup S$.**

**(d) True. If $\sup S \in S$ then it is the maximum element because it is an upper bound, so $\sup S \geq s$ for all $s \in S$.**

**Conversely if there exists $m = \max S$ then $m$ is an upper bound, and given any other upper bound $M$, $M \geq m$ by definition of upper bound because $m \in S$. Therefore $m$ is the least upper bound, i.e. $\sup S = m \in S$.**

**(e) False. E.g. $S = \{0\}$ has $\sup S = 0$ but $\mathbb{R}\backslash S$ is not even bounded below so has no infimum.**

**(f) True. If $S = (-\infty, a)$ then $\sup S = a$ while $\mathbb{R}\backslash S = [a, \infty)$ so $\inf S^c = a$ also.**

**If $S = (-\infty, a]$ then $\sup S = a$ while $\mathbb{R}\backslash S = (a, \infty)$ so $\inf S^c = a$ also.**

**Conversely, if $\sup S = \inf S^c$ then call this number $a$. Any $x < a$ must be in $S$: if not then $x \in S^c$ but $x < \inf S^c$, a contradiction. Similarly any $x > a$ must be in $S^c$: if not then $x \in S$ but $x > \sup S$, a contradiction.**

**Therefore $(-\infty, a) \subseteq S$ and $(a, \infty) \subseteq S^c$. Finally either $a \in S$ or $a \in S^c$, making $S$ equal to $(-\infty, a]$ or $(-\infty, a)$ respectively.**

3. † **Special long bonus question. Construction of $\mathbb{R}$ from $\mathbb{Q}$.**
Slightly abusing the original notation, say that a subset $S \subset \mathbb{Q}$ is a Dedekind cut if it satisfies (i) and (ii) below.

   (i) If $s \in S$ and $s > t \in \mathbb{Q}$ then $t \in S$ (i.e. $S$ is a semi-infinite interval to the left).

   (ii) $S$ has no maximum.

   (So once we have the reals we'll see that the Dedekind cuts are all of the form $S_r := (-\infty, r) \cap \mathbb{Q}$ for some (any) real number $r$. But we don't know what the reals are in this question!)

   Then we let $\mathbb{R}$ be the set of Dedekind cuts. (I.e. think of identifying $S_r$ with $r \in \mathbb{R}$.)

   Check that we can identify $\mathbb{Q} \subset \mathbb{R}$ by taking $q \in \mathbb{Q}$ to the Dedekind cut $S_q := \{s \in \mathbb{Q} : s < q\}$.

   Define $<$ on $\mathbb{R}$ and show that $\mathbb{R}$ has the completeness property: that any bounded nonempty subset has a least upper bound.

   If you're feeling enthusiastic: for two Dedekind cuts $S_1, S_2$ define their sum $S_1 + S_2 := \{s_1 + s_2 \in \mathbb{Q} : s_1 \in S_1, s_2 \in S_2\}$. Show that this is also a Dedekind cut. Show that this operation $+$ on $\mathbb{R}$ agrees with the usual $+$ on $\mathbb{Q} \subset \mathbb{R}$.

   Similarly define $\times$ on $\mathbb{R}$ and $/$ on $\mathbb{R}\backslash\{0\}$ and show they agree with their standard definitions on $\mathbb{Q}$. Show that $+, \times$ satisfy the usual rules of arithmetic (associative, $\times$ distributes over $+$, $0 + x = x$ and $1 \times x = x$, etc.).

   **This is a long exercise for your entertainment and self-improvement. You may want to think about it over the Christmas break. I am too lazy to write out a full solution but you should know enough by now to know whether or not you're doing it right.**

   **The key part is the completeness property. Suppose $A \subset \mathbb{R}$ is our bounded nonempty subset of $\mathbb{R} := \{$Dedekind cuts$\}$. Then the elements $S$ of $A$ are Dedekind cuts – i.e. subsets of $\mathbb{Q}$ satisfying (i) and (ii). The supremum we want to define is the Dedekind cut given by taking the union of all of these Dedekind cuts:**

$$\sup(A) := \bigcup_{S \in A} S \subset \mathbb{Q}.$$

   **Since $A$ is bounded above this set still satisfies (i) so is indeed a Dedekind cut, i.e. $\sup(A) \in \mathbb{R}$.**

   **For full details see for example W. Rudin, *"Principles of mathematical analysis"*, or the webpage http://tinyurl.com/yjt5olv**

4. † In this question we show that for all $n \in \mathbb{N}$, every positive real number has a unique positive $n$-th root: $\forall n \in \mathbb{N}$, for all $x \in \mathbb{R}, x \geq 0$, there is a unique $y \in \mathbb{R}$, $y \geq 0$, such that $y^n = x$. Prove this in the following steps:

   (i) Prove uniqueness.

   (ii) Prove by elementary means that for all real numbers $a, b$ with $0 < a < b < a + 1$:

$$na^{n-1}(b-a) < b^n - a^n < n(a+1)^{n-1}(b-a)$$

   (Hint: $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$).

(iii) Use (ii) to prove the following for real numbers $x, y$ both $\geq 0$:
If $x^n < y$ then there is $x < x'$ with ${x'}^n < y$;
If $x^n > y$ then there is $x > x'$ with ${x'}^n > y$.

(iv) Now let $y$ be real and $\geq 0$. Show that the set

$$S = \{a \in \mathbb{R} \mid a^n \leq y\}$$

is nonempty and bounded above, hence $x = \sup S$ exists.

(v) Using (iii) and the Lemma about sup proved in the lectures, show that $x^n = y$. (This follows closely the proof given in the lectures for the case $n = 2$ so you should start by reminding yourselves how that works.)

**I am too lazy to write this out in complete detail so I just sketch half of the key steps (ii), (iii). Everything else follows pretty closely what we did in the lectures for the case $n = 2$.**

**For (ii) we have, for example:**

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1}) < n(b-a)(a+1)^{n-1}$$

**and this is half of (ii) (I leave the other half to you).**

**For (iii), suppose that $x^n < y$ and write $x' = x + \varepsilon$; by (ii) we have, if $0 < \varepsilon < 1$:**

$${x'}^n - x^n < n(x+1)^{n-1}\varepsilon$$

**so, by choosing**

$$0 < \varepsilon < \min\left\{1, \frac{y - x^n}{n(x+1)^{n-1}}\right\}$$

**we make sure that ${x'}^n < y$. This is half of (iii) and I leave the other half to you.**

5. Prove that if $x, y, z$ are real numbers such that $x+y+z = 0$, then $xy+yz+zx \leq 0$.

**Expand $0 = (x+y+z)^2$.**

6. Show that any positive periodic decimal expansion is rational, and in fact can be written as

$$p\big/99\ldots9900\ldots00 \quad (m \text{ 9s and } n \text{ 0s})$$

for some integers $p, m, n \geq 0$.
Deduce that any integer divides some number of the form $99\ldots9900\ldots00$.

**Let the decimal expansion be $x = a_0.a_1a_2\ldots a_n(\overline{b_1\ldots b_m})$, where $\overline{\cdot}$ denotes recurring periodically. Then**

$$\begin{aligned} x &= \frac{a_0a_1\ldots a_n}{10^n} + \frac{b_1\ldots b_m}{10^n}(10^{-m} + 10^{-2m} + 10^{-3m} + \ldots) \\ &= \frac{a_0a_1\ldots a_n}{10^n} + \frac{b_1\ldots b_m}{10^n}\frac{1}{10^m - 1} \\ &= \frac{(10^m - 1)a_0a_1\ldots a_n + b_1\ldots b_m}{(10^m - 1)10^n}, \end{aligned}$$

**which is of the form claimed, with $p = (10^m - 1)a_0a_1\ldots a_n + b_1\ldots b_m$.**

**As proved in lectures, $x = 1/q$ has periodic decimal expansion since it is rational. Therefore we get $1/q = p/99\ldots9900\ldots00$ for some intreger $p$, and thus $99\ldots9900\ldots00/q = p$ as required.**

7. Show by using decimal expansions that between any two distinct real numbers there exists a rational number and an irrational number. (Last week you showed this by a different method.)

**Write the smaller number as $A = a.a_1a_2\ldots a_ka_{k+1}\ldots$ and the bigger as $B = a.a_1a_2\ldots b_kb_{k+1}\ldots$, differing for the first time in the $k$th place.**

**As in lectures can assume that neither ends in an infinite string of 9s. Choose the first digit $a_{k+i},\ i > 0$ after $a_k$ that is not a 9; setting it to 9 and all later digits $a_{k+j},\ j > i$ to zero gives a rational number $C$ between $A$ and $B$. Now find an irrational number $C$ between $A$ and $B$.**

*You should prepare starred questions * to discuss with your personal tutor.*
*Questions marked † are slightly harder (closer to exam standard), but good for you.*