M1F Foundations of Analysis — Problem Sheet 7

1. How many monomials in x, y are there of degree $\leq k$? (Justify your answer.)

   (The monomials of degree k are the polynomials with one term $x^i y^j$ with $i + j = k$ and coefficient 1. Here i, j, k are positive but can be 0.)

   The number of monomials in x, y of degree $\leq k$ is $(k^2 + 3k + 2)/2$. You can prove this by induction on k:

   True for $k = 0$: the only monomial is 1 and the formula gives $2/2 = 1$.

   Suppose true for k. Then the monomials of degree $\leq k + 1$ are the monomials of degree $\leq k$ plus x^{k+1}, $x^k y$, $\ldots$, xy^k, y^{k+1}. There are $k + 2$ of these so we find the total $(k^2 + 3k + 2)/2 + k + 2 = (k^2 + 5k + 6)/2 = ((k + 1)^2 + 3(k + 1) + 2)/2$ as required.

2. What is the smallest number in the set $\{x \colon x > 0\}$? Justify your answer. What about $\{x \in \mathbb{Q} \colon x \geq \sqrt{2}\}$?

   There is no smallest number in $\{x \colon x > 0\}$. The number 0 is not in the set, and any number $x > 0$ cannot be the smallest number because there is always a smaller number in the set, such as $x/2$.

   Similarly there is no smallest number in $\{x \in \mathbb{Q} \colon x \geq \sqrt{2}\}$. $\sqrt{2}$ is not in the set. So any smallest number x is $> \sqrt{2}$, so there is a rational number between x and $\sqrt{2}$ by Q5.

3. * Write down a careful proof that for any two *positive* numbers x, y, their mean (or average, or "arithmetic mean") $\frac{1}{2}(x + y)$ is greater than or equal to their "geometric mean" $\sqrt{xy}$:

   $$\frac{x + y}{2} \geq \sqrt{xy}.$$

   When are they equal ?

   BE VERY CAREFUL WITH THIS PROOF: ARE YOUR IMPLICATIONS IN THE CORRECT DIRECTION ?!

   Suppose for a contradiction that it is not true, i.e. $x + y < 2\sqrt{xy}$.

   Squaring both sides, $x^2 + 2xy + y^2 < 4xy \Rightarrow x^2 - 2xy + y^2 < 0 \Rightarrow (x - y)^2 < 0$. Contradiction.

   Equality holds iff it holds at every step of the above calculation, iff $(x - y)^2 = 0$. That is, if and only if $x = y$.

4. Prove that for every positive integer $n \neq 3$, the number $\sqrt{n} - \sqrt{3}$ is irrational.

   Suppose $\sqrt{n} - \sqrt{3} = r$ is rational.

   Write as $\sqrt{n} = r + \sqrt{3}$ and square to give $n = r^2 + 2r\sqrt{3} + 3$.

   So either $r = 0$ (impossible; $n \neq 3$) or $\sqrt{3} = \frac{n - r^2 - 3}{2r}$. But this is rational, a contradiction.

5. Let a be a real number. Prove carefully that for all real numbers $\varepsilon > 0$ there is a rational number r such that $a < r < a + \varepsilon$. (Hint: use the Archimedean Axiom.)

   First choose an integer $q > 0$ such that $q\varepsilon > 2$. Then let $p = \lceil qa \rceil + 1$. (If $\alpha \in \mathbb{R}$, $\lceil \alpha \rceil$ is the round up of α, that is $\lceil \alpha \rceil = \min\{k \in \mathbb{N} \mid k \geq \alpha\}$. It follows that $\alpha \leq]\alpha[< \alpha + 1$.) Then

   $$qa < p < qa + 2 < qa + q\varepsilon$$

   hence $a < p/q < a + \varepsilon$.

6. Show that between any two distinct real numbers there exists a rational number and an irrational number.

The first part of the question is Q5. For the second part, let $a < b$ be real numbers. We look for an irrational number of the form $r\sqrt{2}$, with $r \in \mathbb{Q}$, such that

$$a < r\sqrt{2} < b$$

but this is the same as finding a rational number r such that $\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$ and this is Q5 again.

7. † In this question we construct $\mathbb{Z}$ from $\mathbb{N}$.

   Let $S = \mathbb{N} \times \mathbb{N}$ and define a relation $\sim$ on S by declaring $(m_1, n_1) \sim (m_2, n_2)$ if and only if $m_1 + n_2 = m_2 + n_1$. (Intuitively, (m, n) "is" $m - n$.)

   (i) Prove that $\sim$ is an equivalence relation.

   (ii) Show that $(m_1, n_1) \sim (m_2, n_2)$ if and only if: $\forall p \in \mathbb{N}$, $(m_1 + p, n_1 + p) = (m_2 + p, n_2 + p)$.

   (iii) Show that,$\forall p \in \mathbb{N}$, if $(m_1, p) \sim (m_2, p)$ then $m_1 = m_2$. Similarly, $\forall p \in \mathbb{N}$, if $(p, n_1) \sim (p, n_2)$, then $n_1 = n_2$.

   Denote by $X = S/\sim$ the quotient set.

   Define an operation $\oplus$ on S by declaring $(m, n) \oplus (p, q) = (m + p, n + q)$. Show that

   (iv) If $(m_1, n_1) \sim (m_2, n_2)$ and $(p_1, q_1) \sim (p_2, q_2)$, then $(m_1, n_1) \oplus (p_1, q_1) \sim (m_2, n_2) \oplus (p_2, q_2)$,

   and hence conclude that $\oplus$ defines an operation, denoted $+$, on the quotient set $X = S/\sim$.

   (v) Define a function $i \colon \mathbb{N} \to X$ by declaring, for all $m \in \mathbb{N}$, $i(m) = (m, 0)$. Show that i is injective. By means of i we think of $\mathbb{N}$ as a subset of X. Show that $i(m + p) = i(m) + i(p)$.

   (vi) Write $0 = [(0, 0)] \in X$ (where as usual $[a]$ denotes the equivalence class of a). Show that for all $x \in X$, $x + 0 = x$.

   (vii) Show that for all $x \in X$ there is $y \in X$ such that $x + y = 0$. We write $y = -x$.

   (viii) Show that for all $x \in X$, either $x \in \mathbb{N}$ or $-x \in \mathbb{N}$, and both are true if and only if $x = 0$.

   (ix) Define a bijective function $f \colon \mathbb{Z} \to X$ and show that for all $a, b \in \mathbb{Z}$, $f(a + b) = f(a) + f(b)$.

Most of the question is a "routine" verification that I omit.

For (vii) note that $(n, m) \oplus (m, n) = (n + m, m + n) \sim (0, 0)$ hence $[(m, n)] + [(n, m)] = 0$.

For (viii), either $n \leq m$, in which case $(m, n) = (m - n, 0)$ and $[(m, n)] = i(m - n)$, or $n \geq m$, in which case $(m, n) = (0, n - m)$ and $-[(m, n)] = [(n, m)] = i(n - m)$. Both are true if and only if $m = n$, in which case $(m, n) \sim (0, 0)$, that is, $[(m, n)] = 0$.

For (ix) you want to define

$$f(a) = \begin{cases} (a, 0) & \text{if a} \geq 0, \\ (0, -a) & \text{if a} \leq 0 \end{cases}$$

and prove the statement by a division in four cases: $a > 0, b > 0$; $a > 0, b < 0$, $a < 0, b > 0$, $a, b < 0$.

*You should prepare starred questions * to discuss with your personal tutor.*
Questions marked † are slightly harder (closer to exam standard), but good for you.