M1F Foundations of Analysis **Problem Sheet 6b**

This is a corrected version of sheet 6. In the previous version I accidentally repeated 3 questions from sheet 5: I am sorry about that. I have removed those and added three new questions. I have also slightly edited the last question.

1. Suppose that $f\colon S \to T$ is surjective. Prove that there exists $g\colon T \to S$ such that $f \circ g = \mathrm{id}_T$.

   Given any $t \in T$, there exists $s_t \in S$ such that $f(s_t) = t$ (because f is onto). Define $g(t) = s_t$. Then $f \circ g(t) = f(s_t) = t$ for all $t \in T$, so $f \circ g = \mathrm{id}_T$.

2. * Suppose given functions $h\colon A \to B$ and $g\colon C \to B$. Show that there exists $f\colon A \to C$ such that $h = g \circ f$ if and only if image $(h) \subseteq$ image (g).

   Suppose f exists and pick an element $h(a)$ of image (h). Then it equals $g(f(a))$ so it's in the image of g, as required.

   Conversely suppose image $(h) \subseteq$ image (g). We want to define $f(a) \in C$ for any $a \in A$. Well $h(a) \in$ image $(h) \subseteq$ image (g) so there exists $c \in C$ such that $h(a) = g(c)$. Pick one such c, call it c_a and defined $f(a) := c_a$. This defines a singly valued function f (because I picked only one such c).

   Then for any $a \in A$ we have $g \circ f(a) = g(c_a) = h(a)$ by the definition of c_a. Therefore $g \circ f = h$.

3. † Prove that a function $f\colon B \to C$ is injective if and only if the following statement holds:
   ($\star$) for all sets A and all $g_1, g_2\colon A \to B$, we have $f \circ g_1 = f \circ g_2 \implies g_1 = g_2$.

   Suppose that f is injective and that $f \circ g_1 = f \circ g_2$. Then for any $a \in A$, we have $f(g_1(a)) = f(g_2(a))$. But since f is injective this means that $g_1(a) = g_2(a)$. Since this is true for all a, it means $g_1 = g_2$. Therefore ($\star$) holds.

   Conversely, suppose that ($\star$) holds. Pick $b_1, b_2 \in B$, and define $g_i\colon \{1\} \to B$ by $g_1(1) = b_1$ and $g_2(1) = b_2$.

   Notice that $g_1 = g_2$ if and only if $b_1 = b_2$. Therefore ($\star$) says that ($f(b_1) = f(b_2) \Rightarrow b_1 = b_2$). Sine this is true for any $b_1, b_2 \in B$, this is the statement that f is injective.

4. Let A, B, and C be sets. Suppose given functions $f\colon A \to B$ and $h\colon A \to C$. Show that there exists a function $g\colon B \to C$ such that $g \circ f = h$ if and only if the following condition holds:
   $\forall\, a_1, a_2 \in A,\ f(a_1) = f(a_2) \Rightarrow h(a_1) = h(a_2).$

   Suppose there is $g\colon B \to C$ such that $g \circ f = h$. Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Then

   $$h(a_1) = (g \circ f)(a_1) = g\Big(f(a_1)\Big) = g\Big(f(a_2)\Big) = (g \circ f)(a_2) = h(a_2)$$

   Conversely, suppose that the condition stated in the question holds.

   Choose an element $c_0 \in C$ (any element will do). Then define $g\colon B \to C$ as follows. For $b \in B$, if $b \notin$ image(f), then define $g(b) = c_0$. Otherwise, there is $a \in A$ such that $b = f(a)$ and define $g(b) = h(a)$. Now show that this does not depend on the choice of a and thus g is a well-defined function, and show that $g \circ f = h$.

5. Let S be a set and R an equivalence relation on S. Let $f\colon S \to S/R$ be the function that sends every $a \in S$ to the equivalence class $[a]$. Let T be a set and $h\colon S \to T$ be a function. Show that there is a function $g\colon S/R \to T$ such that $g \circ f = h$ if and only if the following condition holds:
   $\forall\, a, b \in S,\ aRb \Rightarrow h(a) = h(b)$ (that is, h is constant on equivalence classes).

   Convince yourself that $f(a) = f(b)$ if and only if aRb. The statement than follows from Q4.

6. This question should "answer" some questions asked by (some) students on quotients by equivalence relations.

   Let S be a set and R an equivalence relation on S. Suppose that every equivalence class $[a]$ has a unique distinguished representative $\overline{a} \in S$.

   Thus $h(a) = \overline{a}$ defines a function from $h: S \to S$ such that for all $a, b \in S$, $h(a) = h(b)$ if and only if aRb. Show that the assignment

$$[a] \mapsto \overline{a}$$

   gives an invertible function from S/R to the set $\{\overline{a} \mid a \in S\} = \text{image(h)} \subset \text{S}$. Thus we can "identify" S/R with image(f).

   Denote, as in Q5, by $f: S \to S/R$ the function that maps a to $[a]$. By Q5 there is a unique function $g: S/R \to S$ such that $g \circ f = h$, and this function maps $[a]$ to $\overline{a}$. It is enough to show that h is injective. (Why?) But indeed if $\overline{a} = \overline{b}$ then aRb and hence $[a] = [b]$.

7. † We are going to find the number of partitions on the set $\{1, 2, \ldots, n\}$. Let this number be B_n (and set $B_0 = 1$). Prove that

$$B_{n+1} = B_n + \binom{n}{1} B_{n-1} + \binom{n}{2} B_{n-2} + \ldots + nB_1 + B_0.$$

   Now let

$$B(q) = \sum_{n=0}^{\infty} \frac{B_n}{n!} q^n.$$

   Assuming (without justification) that $\frac{d}{dq} \sum = \sum \frac{d}{dq}$, that you can reorder infinite sums, etc., show that

$$\frac{dB}{dq} = \sum_{k=0}^{\infty} \frac{B_k}{k!} \sum_{n=k}^{\infty} \frac{q^n}{(n-k)!}.$$

   Deduce that $\frac{dB}{dq} = e^q B(q)$. Recalling that $B(0) = 1$, show that

$$B(q) = e^{e^q - 1}.$$

   The element $n+1$ can be partitioned into the same equivalence class as any i elements of $\{1, 2, \ldots, n\}$, for any $i = 0, 1, \ldots n$. The number of choices of such elements is $\binom{n}{i}$.

   Then the remaining $(n-i)$ elements of $\{1, 2, \ldots, n\}$ can be partitioned in any way you like – there are B_{n-i} ways of doing this.

   So the number of partitions of $\{1, 2, \ldots, n, n+1\}$ is $\sum_{i=0}^{n} \binom{n}{i} B_{n-i}$, as required.

   Letting $k = n - i$ this can be written $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ (*).

   $dB/dq = \sum_{n=1}^{\infty} \frac{B_n}{n!} d(q^n)/dq = \sum_{n=1}^{\infty} \frac{B_n}{(n-1)!} q^{n-1} = \sum_{n=0}^{\infty} \frac{B_{n+1}}{n!} q^n$.

   Substituting (*) into this gives $\sum_{n=0}^{\infty} \sum_{k=0}^{n} B_k \frac{q^n}{k!(n-k)!}$, which we reorder as $\sum_{k=0}^{\infty} \frac{B_k}{k!} \sum_{n=k}^{\infty} \frac{q^n}{(n-k)!}$, as required.

   Now $\sum_{n=k}^{\infty} \frac{q^n}{(n-k)!} = q^k \sum_{n=k}^{\infty} \frac{q^{n-k}}{(n-k)!} = q^k e^q$, so we have shown that $dB/dq = \sum_{k=0}^{\infty} \frac{B_k}{k!} q^k e^q = B(q) e^q$.

   Therefore $d \log B(q)/dq = e^q$ and so $\log B(q) = e^q + C$. Setting $q = 0$ shows that $C = -1$, so that $B(q) = \exp(e^q - 1)$.