M1F Foundations of Analysis Problem Sheet 3

1. Show *by induction* that $7^n - 3^n$ is always divisible by 4.

   † Can you see the one-line proof that this true, not using induction ?

   Clearly true for $n = 0$. Assume true for $n = k$, then

   $$7^{k+1} - 3^{k+1} = 7(7^k - 3^k) + 7.3^k - 3^{k+1} = 7(7^k - 3^k) + 4.3^k$$

   is also divisible by 4. Therefore true for $n = k + 1$, which completes the induction step.

   The quick way is to notice that $p(x) = x^n - a^n$ has a root at $x = a$, i.e. $p(a) = 0$ so $p(x)$ is divisible by $(x - a)$ (we'll prove this later in the course). Or, explicitly,

   $$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + \ldots + a^{n-2}x + a^{n-1}).$$

   Substituting $x = 7, a = 3$ gives the result.

2. On Mars all months have exactly 30 days. Much neater. Show that some months start on a Monday.

   30 and 7 are coprime, so there exist integers p, q such that $30p + 7q = 1$.

   In fact by Euclid we can work this out as, for instance, $-30.3 + 7.13 = 1$.

   Therefore after 3 months, 13 weeks minus one day have passed, so the day of the week has decreased by one. Therefore eventually all days of the week feature at the start of a month.

3. Professor Corti likes to have fun every 5 days and eat salad every 8 days. Show that he sometimes has fun eating salad.
   After a fun salad, how long is it until his next fun salad ?

   Since 5, 8 are coprime, by Euclid there exist integers p, q such that $5p + 8q = 1$. Eg $p = 5,\ q = -3$ will do (i.e. $5.5 = 8.3 + 1$).

   Fix a fun day. After 5 more fun days, $5.5 = 8.3+1$ days will have passed, i.e. 3 more salad days *plus* one more day. So we are one further day from the last salad day. Repeating we will eventually have a fun day which is 8 days from the last salad day, so it will be a fun salad day.

   Fix a fun salad day. After n more fun days, $5n$ days have passed; if this is another salad day then $5n$ is a multiple of 8.

   But 5 and 8 are coprime, therefore n is also a multiple of 8. Therefore the smallest n can be is 8. Therefore it is 40 days before salad becomes fun again.

4. The pencils in Professor Corti's pencil case have been taken out and laid end to end, exactly reaching the moon 8^{88}cm away. One is shorter than the others; the rest are all exactly 18 cm long. What's the length of the shorter one ?

   Working mod 18 at all times, we have:

   $8^2 \equiv 64 \equiv 10.\ \Rightarrow 8^4 \equiv 100 \equiv 10.\ \Rightarrow 8^8 \equiv 100 \equiv 10.\ \Rightarrow 8^{16} \equiv 100 \equiv 10.\ \Rightarrow 8^{32} \equiv 100 \equiv 10.\ \Rightarrow 8^{64} \equiv 100 \equiv 10.$
   $\Rightarrow 8^{88} = 8^{64+16+8} \equiv 10.10.10 \equiv 10.$

   $\Rightarrow 8^{88} = 18m + 10$ for some m equal to the number of long pencil crayons in the pencil case. So the short one has length 10cm.

5. † At home, Professor Corti keeps an infinite collection of 5 pence and 11 pence coins. Show that he can buy anything that costs 40 pence or more with the correct change.

   This proof is not meant to be comprehensible. You have to see how to make 40 – from eight 5 pences – then use $11-2.5 = 1$ to make 41,42,43,44. Now 45 can be made from nine 5 pences, and we start again. Then you get the general idea. Then you turn it into the abstract incomprehensible proof that follows:

5 and 11 are coprime, and by Euclid (or inspection) we find that $11 - 2.5 = 1$.

Given any number $n \geq 40$, write it as $5q + r$ where $q \geq 8$ and $0 \leq r \leq 4$.

Then $n = 5q + r = 5q + (11 - 2.5)r = 5(q - 2r) + 11r$.

Since $q \geq 8$ and $r \leq 4$, we have that $(q - 2r) \geq 0$. And $r \geq 0$ too.

Therefore he can use $q - 2r$ of his 5p coins and r of his 11p coins to make n pence.

6. $*$ Suppose that m, n are coprime integers that describe some Professor Corti related incident, and that $m|a$ and $n|a$. Show using Euclid's algorithm that $mn|a$.
   Show it again using prime factorisation.
   Show that it need not be true if m and n are not coprime.

   $m|a$ so we can write $a = km$ for some $k \in \mathbb{Z}$.

   $n|a = km$ so from lectures $n|k$ (recall proof: since n, m coprime we can write $mp + nq = 1$, therefore $kmp + knq = k$ and $n|km$ so n divides the RHS).

   $n|k \Rightarrow nm|km$ so $mn|a$.

   Proof using prime factorisation: Write $m = \prod_i p_i$ and $n = \prod_j q_j$ where $\{p_i\}$ and $\{q_j\}$ are completely distinct. Now $m|a$ so $a = km$ for some k so the prime factorisation of a is that of k times that of m. In other words, it contains $\prod_i p_i$. Similarly $n|a$ so it also contains $\prod_j q_j$. Since it is unique and the p_i, q_j are distinct, it must contain $\prod_i p_i \prod_j q_j = mn$. Thus $mn|a$.

   Counterexample when not coprime: set $n = m = a$.

7. † In this question we will give a different proof of the famous consequence of Euclid's algorithm from lectures, without using Euclid.

   Fix positive integers a and b with highest common factor $h := \text{hcf}(a, b)$, and let

$$S := \{x > 0 \colon \exists \lambda, \mu \in \mathbb{Z} \text{ such that } x = \lambda a + \mu b\}.$$

   - Show that S is nonempty and bounded below (so it has a minimum element).
   - Show that $h | \min S$.
   - By dividing $\min(S)$ into a and considering the remainder, show that $\min(S)|a$.
   - Conclude that $h = \min S$ and that $\exists \lambda, \mu \in \mathbb{Z}$ such that $h = \lambda a + \mu b$.

   We prove it point by point as follows.

   - **Choosing $\lambda = 1$, $\mu = 0$ we find $a \in S$ so it's nonempty, and bounded below by 0 (by its definition).**
   - **Since $h|a$ and $h|b$, then h divides every element of S. In particular then, $h| \min S$.**
   - **Write $a = q \min(S) + r$ with $0 \leq r < \min(S)$. If $r > 0$ then it is in S (because $r = a - q \min(S)$ is a linear combination of a, b since $\min(S)$ is). But it is $< \min(S)$, which is a contradiction. So $r = 0$. So $\min(S)|a$.**
   - **Similarly $\min(S)|b$. So $\min(S)$ is a common factor, so it is $\leq h$. But we showed $h| \min(S)$ so $h \leq \min(S)$. So in fact $h = \min(S)$.**

     Therefore $h \in S$, i.e. $\exists \lambda, \mu \in \mathbb{Z}$ such that $h = \lambda a + \mu b$.