M1F Foundations of Analysis — Problem Sheet 2

1. What is the biggest element of the set $\{x \in \mathbb{R} \colon x < 1\}$? Justify your answer carefully.

   It does not exist. Suppose it did, call it $m < 1$. Let $n = (m+1)/2$. Then $m = (m+m)/2 < (m+1)/2 < (1+1)/2 = 1$ shows that $m < n < 1$, so n is a larger element of the set: a contradiction.

2. Let n be an integer. Prove carefully that if n^2 is divisible by 3 then so is n.
   (Hint: any integer can be written in the form $3m$ or $3m+1$ or $3m+2$, for some integer m.)

   Then prove carefully that $\sqrt{3}$ is irrational.

   Suppose that n is *not* divisible by 3. Then n can be written as either:

   Case (i): $n = 3m+1 \;\Rightarrow\; n^2 = 9m^2 + 6m + 1 = 3M + 1$ where $M = 3m^2 + 2m$. So n^2 not divisible by 3.

   Case (ii): $n = 3m+2 \;\Rightarrow\; n^2 = 9m^2 + 12m + 4 = 3M + 1$ where $M = 3m^2 + 4m + 1$. So n^2 not divisible by 3.

   So (n not divisible by 3) $\Rightarrow$ (n^2 not divisible by 3).

   Therefore, conversely, (n^2 divisible by 3) $\Rightarrow$ (n divisible by 3).

   Suppose $\sqrt{3} = p/q$, where p, q are integers with no common factors. Squaring proves that $p^2 = 3q^2$. So p^2 is divisible by 3. So p is also divisible by 3.

   So we can write $p = 3P$. So $9P^2 = 3q^2$, and $q^2 = 3P^2$ is also divisible by 3. Therefore q is also divisible by 3. So p, q have a common factor of 3. Contradiction.

3. Are these deductions correct or not ?

   (a) My dog barks if I get out of bed on the right. I get out of bed on the left. Therefore my dog is silent.

   False. Right $\Rightarrow$ Bark. Negating gives Silent $\Rightarrow$ Left. We are not told if Silent $\Leftarrow$ Left.

   (b) My other dog barks only if I get out of bed on the right. I get out of bed on the left. Therefore he won't bark.

   True. Bark $\Rightarrow$ Right. Therefore Not Right $\Rightarrow$ Not Bark. I.e. Left $\Rightarrow$ Doesn't bark.

4. Prove or disprove the following statements:

   (a) the sum of two irrational numbers is always irrational

   False. $\sqrt{2}$ and $-\sqrt{2}$ are irrational numbers, but their sum is rational.

   (b) the sum of a rational number and an irrational number is always irrational.

   True. Suppose not. Then there exist integers $p, q \neq 0, p', q' \neq 0$ and an irrational number i such that $p/q + i = p'/q'$.

   Rearranging gives $i = (p'q - pq')/qq'$, which is rational. Contradiction.

   (c) if n and k are positive integers, then $n^k - n$ is always divisible by k.

   False. $2^4 - 2 = 14$ is not divisible by 4.

   (d) $\exists \epsilon > 0$ such that $\forall N \in \mathbb{N}\backslash\{0\},\ \forall n \geq N,\ \frac{1}{n} < \epsilon$.

   True: e.g. take $\epsilon = 2$. Since $n \geq N \geq 1$, then $\frac{1}{n} \leq 1 < \epsilon$.

5. † You throw n infinitely long matches (from your infinitely long matchbox) onto the ground. Prove that you divide the ground into at most $\frac{1}{2}(n^2 - 3n + 2)$ *interior* regions. (You may assume without proof that the earth is flat.)

   How can you get equality ?

   True for $n = 1$: a line divides the plane into 0 interior regions (just 2 exterior ones).

Suppose true for $n = k$, and introduce a $(k+1)$th line. This intersects each of the k original lines in at most one point each. So we get at most k intersection points along the new line, dividing the line into at most $k+1$ segments – the first and last exterior segments, and $\leq k-1$ interior segments.

Each interior segment divides a region that existed before into 2 regions. Therefore one extra region is added for each segment. Therefore at most $k-1$ new regions are added.

Therefore the total number of regions is $\leq \frac{1}{2}(k^2-3k+2)+k-1 = \frac{1}{2}(k^2-k) = \frac{1}{2}((k+1)^2-3(k+1)+2)$. So it is also true for $n = k+1$.

The proof shows we get equality if and only if, for all k, the $(k+1)$th line intersects *all* of the previous k lines, and in distinct points. This is true if and only if none of the lines are parallel, and no 3 intersect in the same point.

6. * For which $n \in \mathbb{N}$ is $n! < 2^n$?

   Experimentation seems to show that it is true for precisely $n \leq 3$. By computation it is true for $n \leq 3$, and false for $n = 4$. So by induction it is enough to prove that when $k \geq 4$, if $k! \geq 2^k$ then $(k+1)! \geq 2^{k+1}$.

   So assume $k! \geq 2^k,\ k \geq 4$. Then $(k+1)! = (k+1).k! \geq 5.2^k > 2.2^k = 2^{k+1}$, as required.

7. Show that $1 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

   Clearly true for $n = 1$: $1 = (1.2/2)^2$.

   Suppose true for $n = k$. Then $1+2^3+\ldots+k^3+(k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 = \frac{k^4+2k^3+k^2+4(k^3+3k^2+3k+1)}{4} = \frac{k^4+6k^3+13k^2+12k+4}{4}$.

   But $\left(\frac{(k+1)(k+2)}{2}\right)^2 = \frac{(k^2+3k+2)^2}{4} = \frac{k^4+6k^3+13k^2+12k+4}{4}$. So true also for $n = k+1$.

   So by induction true for all $n \geq 1$.

*You should prepare starred questions * to discuss with your personal tutor.*
Questions marked † are slightly harder (closer to exam standard), but good for you.