M1F Foundations of Analysis — Problem Sheet 1

1. Let A be the set $\{1, 3, -6, \{1, -6\}, \text{Doncaster}, \{1\}, X\}$. Which of the following statements are true and which are false ? (Just write T or F in each case.)

   (a) $X \in A$ **(T)**

   (b) $\{X\} \in A$ **(F)**

   (c) $\{X\} \not\subseteq A$ **(F)**

   (d) $\{1, -6\} \in A$ **(T)**

   (e) $\{1, 3\} \notin A$ **(T)**

   (f) $\{\{1, -6\}\} \subseteq A$ **(T)**

   (g) $\{\text{Doncaster}\} \subseteq A$ **(T)**

   (h) $\{1, -6\} \not\subseteq A$ **(F)**

   (i) $\emptyset \subseteq A$ **(T)**

2. * Describe the following sets. Prove your answers carefully, except in (d) (which we'll cover later).

   (a) $\bigcup_{n=1}^{\infty}(1/n, \infty)$

   (b) $\bigcap_{n=1}^{\infty}(0, 1/n)$

   (c) $\bigcup_{n=1}^{\infty}\{x \in \mathbb{R}\colon -n < x < n\}$

   (d) $\bigcap_{n=1}^{\infty}\{x \in \mathbb{Q}\colon 2 - \frac{1}{n} < x^2 < 2 + \frac{1}{n}\}$

   (a) $\bigcup_{n=1}^{\infty}(1/n, \infty) = (0, \infty)$ **because** $x \in$ **LHS** $\iff x > 1/n$ **for some** $n \geq 1 \iff x > 0 \iff x \in$ **RHS.**

   (b) $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$ **because if** $x \in (0, 1/n)$ $\forall n$ **then** $x < 1/n$ $\forall n \Rightarrow x \leq 0$ **but** $x > 0$ **which is a contradiction.**

   (c) $\bigcup_{n=1}^{\infty}(-n, n) = \mathbb{R}$ **because** $\forall\, x \in \mathbb{R}$, $\exists n \in \mathbb{R}$ **such that** $n > |x|$, **so** $x \in (-n, n)$.

   (d) $\bigcap_{n=1}^{\infty}\{x \in \mathbb{Q}\colon 2 - \frac{1}{n} < x^2 < 2 + \frac{1}{n}\} = \emptyset$.

   Proof: $\sqrt{2} \notin \mathbb{Q}$ **so if** $x \in \mathbb{Q}$ **then** $x^2 \neq 2$.

   So either (i) $x^2 > 2$ **or (ii)** $x^2 < 2$. **In case (i), find** n **such that** $1/n < x^2 - 2$ **(i.e. pick** $n > 1/(x^2 - 2)$**). Then** $x^2 > 2 + 1/n$ **so** $x \notin \{x \in \mathbb{Q}\colon 2 - \frac{1}{n} < x^2 < 2 + \frac{1}{n}\}$ **So** $x \notin \bigcap_{n=1}^{\infty}$. **Similarly for case (ii).**

   Therefore there are no rational numbers in $\bigcap_{n=1}^{\infty}$. **Since** $\bigcap_{n=1}^{\infty} \subset \mathbb{Q}$ **it must be empty.**

3. Which of the following statements are true and which are false ?

   (a) $x^2 - 5x + 6 = 0 \;\Rightarrow\; x = 2$ **(F, x might be 3)**

   (b) $x^2 - 5x + 6 = 0 \;\Leftarrow\; x = 3$ **(T)**

   (c) $x^2 - 5x + 6 = 0 \;\Leftrightarrow\; (x = 2 \text{ or } x = 3)$ **(T)**

   (d) For $x^2 - 5x + 6$ to be zero it is necessary that $x = 3$ **(F, $x = 2$ is ok)**

   (e) If $x^2 - 5x + 6 = 0$ then $x = 3$ **(F, x might be 2)**

   (f) $x = 3$ if $x^2 - 5x + 6 = 0$ **(F, x might be 2)**

   (g) $x = 3$ only if $x^2 - 5x + 6 = 0$ **(T)**

   (h) $x = 1$ if $x^2 - 2x + 1 = 0$ **(T, but only because this equation has just one (double) root)**

4. † Suppose we know that the statement P holds unless Q holds. Which of the following statements follow and which do not ?

(a) P

(b) $Q \Rightarrow P$

(c) $\overline{Q} \Rightarrow P$

(d) $Q \Rightarrow \overline{P}$

The given statement tells us that P holds if Q does not. It tells us *nothing* when Q holds (i.e. P may or may not be true in this case). If you dispute this, think about the example "I will fail the exam unless I do some work". This does not guarantee that I will pass if I do do some work!

So it is precisely the statement $\overline{Q} \Rightarrow P$. So (c) is true, the rest *do not follow* from what we're told (even though in a given case they might be true).

*You should prepare starred questions * to discuss with your personal tutor.*
Questions marked † are slightly harder (closer to exam standard), but good for you.