M1F Foundations of Analysis — Problem Sheet 10

1. * Write $x^{2n+1}-1$ as a product of *real* linear and quadratic factors.

   Write $x^{2n}+x^{2n-1}+\ldots+x+1$ as a product of quadratic real factors.

   Suppose that $n \geq 1$. Let $\omega = e^{2\pi i/(2n+1)}$. Why is $\sum \omega^{i+j} = 0$, where the sum is over all i and j from 1 to $2n+1$ such that $i < j$?

   Roots of $x^{2n+1}=1$ come in conjugate pairs $e^{2k\pi i/(2n+1)}$ and $e^{-2k\pi i/(2n+1)}$ for $k=1,\ldots,n$, plus there is the real root 1 (when $k=0$).

   Therefore $x^{2n+1}-1=(x-\omega)(x-\bar{\omega})\ldots(x-\omega^n)(x-\bar{\omega}^n)(x-1)$ where $\omega = e^{2\pi i/(2n+1)}$.

   That is, $x^{2n+1}-1$ factors as $(x^2-2\cos(\frac{2\pi}{2n+1})x+1)\ldots(x^2-2\cos(\frac{2n\pi}{2n+1})x+1)(x-1)$.

   Now $x^{2n+1}-1=(x-1)(x^{2n}+x^{2n-1}+\ldots+x+1)$ so from the above factorisation we find that $x^{2n}+x^{2n-1}+\ldots+x+1=(x^2-2\cos(\frac{2\pi}{2n+1})x+1)\ldots(x^2-2\cos(\frac{2n\pi}{2n+1})x+1)$.

   The sum of the products of pairs of roots ω^i, ω^j of the equation $x^{2n+1}-1=0$ equals the coefficient of x^{2n-1} in $x^{2n+1}-1=0$, which is zero.

2. † Suppose that

$$\begin{aligned} a+b+c &= 3, \\ a^2+b^2+c^2 &= 3, \\ a^3+b^3+c^3 &= 3. \end{aligned}$$

   Find a cubic equation whose roots are a, b, c and hence solve these equations.

   $(a+b+c)^2=(a^2+b^2+c^2)+2(ab+ac+bc)$, so $ab+ac+bc=3$.

   $(a+b+c)^3=(a^3+b^3+c^3)+3(a^2b+a^2c+ab^2+b^2c+ac^2+bc^2)+6abc$, and $(a^2+b^2+c^2)(a+b+c)=(a^3+b^3+c^3)+(a^2b+a^2c+ab^2+b^2c+ac^2+bc^2)$. Putting this all together shows that $abc=1$.

   Therefore the cubic with these roots $x^3-(a+b+c)x^2+(ab+ac+bc)x-(abc)=0$ is $x^3-3x^2+3x-1=0$.

   Either notice that this has a root $x=1$ and so factorise as $(x-1)(x^2-2x+1)=0$, or notice straight away that it is $(x-1)^3=0$.

   Therefore all 3 roots are 1; therefore a, b, c are all 1.

3. Find an exact value for $\cos\frac{\pi}{24}$. Hence explicitly factorise the polynomial $x^{48}-1$ in $\mathbb{R}[x]$.

   I am too lazy to do all of this. For the first part of the question consider that $\cos\frac{x}{2}=\sqrt{\frac{1+\cos x}{2}}$, which then you can use to show that $\cos\frac{\pi}{24}=\frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}$. For the second part of the question you should implelment Q1.

4. Show that $\cos\frac{2\pi}{9}$ is a root of the cubic equation

$$8x^3-6x+1=0$$

   Thinking about the other roots, deduce that

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}=0$$

   and

$$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}=-\frac{1}{8}$$

   I will do this in class on the last day, hopefully.

5. Find a degree 3 polynomial equation with rational coefficients having $x = \cos\frac{2\pi}{7}$ as a root. Use the cubic formula to write down a formula for $\cos\frac{2\pi}{7}$. Stare at your formula and see what you can learn from it.

   Again, I am too lazy to do this. You should get the equation $8x^3 + 4x^2 - 4x - 1 = 0$ for $x = \cos x$. When you write the final formula you will be stuck with the cubic root of a complex number. From here no further progress is possible.

6. Find the highest common factor of the polynomials $A(x) = x^4 + x + 1$, $B(x) = x^3 + 2x^2 + 2x + 1$ in $\mathbb{Q}[x]$. If $C(x)$ is this highest common factor, find polynomials $P(x)$, $Q(x)$ in $\mathbb{Q}[x]$ such that $A(x)P(x) + B(x)Q(x) = C(x)$.

   Hind: use long division for polynomials and imitate what you do in $\mathbb{Z}$.

*You should prepare starred questions * to discuss with your personal tutor.*
Questions marked † are slightly harder (closer to exam standard), but good for you.