KMB, 13/12/01.

M1F Foundations of Analysis—Problem Sheet 11, hints and solutions.

1)

(i) Say $a, b \in X$ and $f(a) = f(b)$. Then $g(f(a)) = g(f(b))$ so $(g \circ f)(a) = (g \circ f)(b)$ and hence $a = b$, and because a and b were arbitrary we deduce that f is injective.

(ii) [typo in the question—should say "surjective".] Say $y \in Y$. Define $x = g(y)$. Then $f(x) = f(g(y)) = (f \circ g)(y) = y$ and because y was arbitrary we see that f is surjective.

(iii) This is immediate from (i) and (ii).

2†)

a) A is countable so let's assume $A = \mathbb{N}$. Say $f : \mathbb{N} \to B$ is a surjection. Define a map $g : B \to \mathbb{N}$ by letting $g(b)$ be the smallest $n \in \mathbb{N}$ such that $f(n) = b$. Note that by surjectivity such b must exist. Then g is easily checked to be an injection, so we can regard B as a subset of $\mathbb{N}$. If B is not finite then we can count it as $B = \{b_1, b_2, b_3, \ldots\}$ where b_1 is the smallest element of B, b_2 is the second smallest, and so on. So B is countable.

b) Fix bijections $f_n : N \to A_n$. Then there's a surjection $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{n \in \mathbb{N}} A_n$ defined by $f(m, n) = f_n(m)$. Hence by a) we see that the union is either finite or countable. But the union contains A_1 which is countable and hence infinite, and so the union must be countable.

c) Fix $n \in \mathbb{N}$ and let A_n be the set of subsets of $\mathbb{N}$ of size at most n. There's a surjection $\mathbb{N}^n \to A_n$ defined by $f(a_1, a_2, \ldots, a_n) = \{a_1, a_2, \ldots, a_n\}$. Hence A_n is countable and so by b) we see that the set of finite subsets of $\mathbb{N}$ is also countable. Alternatively define an injection from the finite subsets of $\mathbb{N}$ to $\mathbb{N}$ sending $\{a_1, a_2, \ldots, a_n\}$ (assumed to be in increasing order) to $2^{a_1} 3^{a_2} 5^{a_3} \ldots p_n^{a_n}$, where p_i is the ith prime.

d) If the set of infinite subsets of $\mathbb{N}$ were also countable, then the set of all subsets of $\mathbb{N}$ would be countable by c), and this contradicts Cantor's diagonal argument: the power set of $\mathbb{N}$ is not countable.

e) The set of all subsets of $\mathbb{N}$ is uncountable. If T is any such subset, then consider the following equivalence relation on $\mathbb{N}$: If $n = 2r - 1$ is odd then $n \sim n$, and if $r \in T$ then also $n \sim n + 1$. But n is not related to any other natural number. Similarly, if $n = 2r$ is even, then $n \sim n$, and if $r \in T$ then also $n \sim n - 1$, but n is related to no other natural number. Hence the equivalence classes look like $\{2r - 1\}, \{2r\}$ if $r \notin T$ and $\{2r - 1, 2r\}$ if $r \in T$, as r runs through $\mathbb{N}$. The resulting equivalence relations are all distinct and so because there are uncountably many choices for T, there are also uncountably many equivalence relations on $\mathbb{N}$.

f) The set of functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable, as it is naturally in bijection with the set of subsets of $\mathbb{N}$. Hence the set of functions $f : \mathbb{N} \to \mathbb{N}$ is also uncountable.

3)

(i) $\binom{p}{a} = \frac{p!}{a!(p-a)!}$ and the LHS is an integer, so the RHS must be too, after cancelling. But after all the cancelling has finished on the RHS, the prime p

will still be left in the numerator because both a and $p-a$ are strictly smaller than p. So p must divide the RHS so p divides the LHS.

(ii) This follows instantly from the binomial theorem and the definition of "mod".

(iii) It's true for $n=0$ and $n=1$. For general $n>0$ one can easily prove $n^p \equiv n \bmod p$ by induction on n, and for $n<0$ one can either do induction on $-n$ or deal with $p=2$ explicitly and then for general p use the fact that $(-m)^p = -m^p$ to deduce the negative case from the positive case.

(iv) If n is prime to p then there is λ and μ such that $\lambda n + \mu p = 1$. We know $n^p \equiv n \bmod p$ and now multiplying both sides by λ, we deduce that $n^{p-1} \equiv 1 \bmod p$.

4)

(i) By the binomial theorem it's $\binom{17}{15} = \frac{17.16}{2} = 17.8 = 136$.

(ii) It's $\binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3y + \binom{4}{2}(2x)^2y^2 + \binom{4}{3}(2x)y^3 + \binom{4}{4}y^4 = 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$.

5)

(i) By the multinomial theorem it's $\binom{7}{2,2,3} = \frac{7!}{2!2!3!} = \frac{7!}{2.2.6} = 1.2.3.5.7 = 21.10 = 210$.

(ii) By the multinomial theorem the product is a sum of things of the form $\binom{5}{a,b,c}1^a x^b (x^3)^c$ and we want the degree of this to be 11 so $b+3c=11$ and $a+b+c=5$. Subtracting, we see $2c-a=6$ and hence $2c \geq 6$ meaning that $c=3,4,5$. Then $2c-a=6$ gives $a=0,1,2$ and only $c=3$ and $a=0$ gives a non-negative value of b, namely $b=2$. So the answer is $\binom{5}{0,2,3} = \frac{5!}{2!3!} = 4.5/2 = 10$.