KMB, 13/12/01

M1F Foundations of analysis—Problem Sheet 11.

1) Let $f : X \to Y$ be a function, and let $g : Y \to X$ be a function.

(i) If $g \circ f = id_X$ then prove that f is injective.

(ii) If $f \circ g = id_Y$ then prove that f is surjective.

(iii) Deduce that if g is an inverse function for f (that is, if both $g \circ f$ and $f \circ g$ are the identity function), then f is a bijection. Remark: this is frequently the easiest way to prove that a function is a bijection—just write down an inverse.

2†) For the experts! Prove the following.

a) If A is countable and there's a surjection $A \to B$ then B is either finite or countable.

b) If A_1, A_2, $A_3 \ldots$ are countable, then so is $\bigcup_{n \in \mathbb{N}} A_n$.

c) The set of finite subsets of $\mathbb{N}$ is countable.

d) The set of infinite subsets of $\mathbb{N}$ is uncountable.

e) The number of equivalence relations on $\mathbb{N}$ is uncountable.

f) The set of functions $f : \mathbb{N} \to \mathbb{N}$ is uncountable.

3) Here is the proof of Proposition 6.14 in the course, a.k.a. Fermat's Little Theorem.

(i) Prove that if p is prime and $1 \leq a \leq p-1$ then $\binom{p}{a}$ is a multiple of p.

(ii) Deduce that if $n \in \mathbb{Z}$ then $(1+n)^p \equiv 1 + n^p \bmod p$.

(iii) Prove that if $n \in \mathbb{Z}$ then $n^p \equiv n \bmod p$.

(iv) Prove that if $n \in \mathbb{Z}$ is prime to p then $n^{p-1} \equiv 1 \bmod p$.

4)

(i) Find the coefficient of x^{15} in $(1+x)^{17}$.

(ii) Expand $(2x+y)^4$.

5)

(i) Find the coefficient of $x^2y^2z^3$ in $(x+y+z)^7$.

(ii) Find the coefficient of x^{11} in $(1+x+x^3)^5$.