KMB, 6/12/01

M1F Foundations of Analysis—Problem Sheet 10, hints and solutions.

1)

(i) is true: $x \in A \cap (B - C)$ if and only if $x \in A$ and $x \in B - C$, if and only if $x \in A$, $x \in B$ and $x \notin C$. On the other hand, $x \in (A \cap B) - (A \cap C)$ iff $x \in A \cap B$ and $x \notin A \cap C$, iff $x \in A$, $x \in B$, and $x \notin A \cap C$, which also happens iff $x \in A$, $x \in B$ and $x \notin C$. It would be much easier just to draw a picture, I guess!

(ii) is not true, because $A - (B - C) \subseteq A$ but C is a subset of the RHS and there is no reason to suspect that $C \subseteq A$ in general. For example, if $A = B = \emptyset$ and $C = \{1\}$ then this will give a counterexample.

(iii) This one is true: if one draws the diagrams then one sees that both sets are just the union of A and $\{x : x \in B \cap C, x \notin A\}$.

2) We see $A_1 \supseteq A_2 \supseteq \ldots$, so $A_1 \cap A_2 = A_2$ and $\cap_{n=1}^{10} A_n = A_{10}$. On the other hand, $\cap_{n=1}^{\infty} A_n$ is $\{x \in \mathbb{R} : -1/n < x < 1/n$ for all $n \in \mathbb{N}\}$ and this is just $\{0\}$. The reason for this is that if $x > 0$ then for sufficiently large n we will have $1/n < x$, and similarly if $x < 0$ then for sufficiently large n we will have $-1/n > x$.

3)

a) f is bijective: if $f(x) = f(y)$ then $3 - 2x = 3 - 2y$ so $2x = 2y$ so $x = y$. On the other hand, if t is any real then $t = f(s)$ for $s = \frac{1}{2}(3 - t)$.

b) This is neither injective nor surjective, as $f(3) = f(-3) = 81$ and on the other hand -1 is not $f(x)$ for any x.

c) This function is both injective and surjective. It's injective because if $x^4 = y^4$ and x, y are positive, then $x = y$ (as $x < y$ implies $x^4 < y^4$ and $x > y$ implies $x^4 > y^4$). On the other hand, every positive real has a unique positive real fourth root, so the function is also surjective.

d) This function is still injective, as $3 - 2x = 3 - 2y$ implies $x = y$ by an easy calculation, but it is not surjective, as, for example, there is no $s \in S$ such that $f(s) = 0$ (s would have to be 1.5 which is not an integer).

e) This function is not surjective (because there is certainly no x such that $f(x) = -1$, as $f(x)$ is the square of something) but it is injective, because if $(x + \sqrt{2})^2 = (y + \sqrt{2})^2$ then either $x + \sqrt{2} = y + \sqrt{2}$ (in which case $x = y$) or $x + \sqrt{2} = -\sqrt{2} - y$, which simplifies to $x + y = -2\sqrt{2}$, a contradiction as one side is rational and the other irrational.

4)

a) Say $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$ by definition. So $f(x) = f(y)$ by injectivity of g. So $x = y$ by injectivity of f. But x and y were arbitrary, so $(g \circ f)$ is injective.

b) If $c \in C$ then by surjectivity of g there is b in B such that $g(b) = c$. Now by surjectivity of f there is $a \in A$ such that $f(a) = b$. Hence $(g \circ f)(a) = c$. But c was arbitrary, so $(g \circ f)$ is surjective.

c) This follows immediately from parts a) and b).

5) We are given that g and h are inverses of f. So, by definition, $f \circ g = f \circ h = id_T$. So if $t \in T$ we have $f(g(t)) = t = f(h(t))$. But f is injective, so $g(t) = h(t)$.

6)

a) One can use the same trick that we used for counting the positive rationals: if $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ then

$$A \times B = \{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3), (a_4, b_1) \ldots\},$$

counting the pairs (a_i, b_j) for which $i + j = n$ and slowly letting n increase.

b) We know $\mathbb{Z}$ is countable, from lectures. So the question is easy by a) and induction on n, as $\mathbb{Z}^n = \mathbb{Z}^{n-1} \times \mathbb{Z}$.