KMB, 06/12/01

M1F Foundations of analysis—Problem Sheet 10.

1) Let A, B, C be sets. Proofs or counterexamples required for the following:
(i) $A \cap (B - C) = (A \cap B) - (A \cap C)$.
(ii) $A - (B - C) = (A - B) \cup C$.
(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Note that this one is the "opposite" of Proposition 8.1.

2) For $n \in \mathbb{N}$ define A_n to be the set $\{x \in \mathbb{R} : -1/n < x < 1/n\}$. Work out $A_1 \cap A_2$. Work out $\cap_{n=1}^{10} A_n$. Work out $\cap_{n=1}^{\infty} A_n$.

3) For each of the functions $f : S \to T$ below, establish (with proof) whether or not f is injective and whether or not f is surjective.
a) $S = T = \mathbb{R}$, $f(x) = 3 - 2x$
b) $S = T = \mathbb{R}$, $f(x) = x^4$
c) $S = T = \{x \in \mathbb{R} : x > 0\}$, $f(x) = x^4$
d) $S = \mathbb{Z}$, $T = \mathbb{R}$, $f(x) = 3 - 2x$
e) $S = \mathbb{Q}$, $T = \mathbb{R}$, $f(x) = (x + \sqrt{2})^2$.

4) Let A, B, C be sets, and $f : A \to B$ and $g : B \to C$ be functions.
a) Prove that if f and g are injective, then $g \circ f$ is injective.
b) Prove that if f and g are surjective, then $g \circ f$ is surjective.
c) Prove that if f and g are bijective, then $g \circ f$ is bijective.

5) Prove that the inverse of a bijection $f : S \to T$ is unique. That is, if $f : S \to T$ is a bijection, and we have two functions $g : T \to S$ and $h : T \to S$ such that both g and h are inverses of f, then $g(t) = h(t)$ for all $t \in T$.

6)
(i) Prove that if A and B are countable sets, then $A \times B$ is also countable.
(ii) Prove that for any $n \in \mathbb{N}$, the set $\mathbb{Z}^n$ is countable.