KMB, 30/11/01

M1F Foundations of Analysis—Problem Sheet 9, hints and solutions.

1*)

(i)

a) If $x \in \mathbb{R}$ then $|x - x| = 0 < 1$ so $\sim$ is reflexive. If $x, y \in \mathbb{R}$ then $|x - y| < 1 \Rightarrow |y - x| = |x - y| < 1$ so $\sim$ is symmetric. However $a = 0, b = 0.9$ and $c = 1.8$ show that $\sim$ is not transitive: we have $a \sim b$ and $b \sim c$ but $a \not\sim c$.

b) $\sim$ is easily checked to be reflexive and transitive. But it's not symmetric because $1 \sim 2$ and $2 \not\sim 1$.

c) $1+1$ is not a multiple of 3, so $1 \not\sim 1$ and $\sim$ is not reflexive. It is symmetric because $x + y = y + x$. Finally it's not transitive because $1 \sim 2$ and $2 \sim 1$ but $1 \not\sim 1$.

d) 0 is rational so $\sim$ is reflexive. If $x - y$ is rational then so is $y - x$ so $\sim$ is symmetric. Finally if $x \sim y$ and $y \sim z$ then both $x - y$ and $y - z$ are rational, so their sum, which is $x - z$, is too. Hence $x \sim z$ and $\sim$ is transitive.

0.5 marks per property, giving 1.5 marks for each of the four relations, that is, 6 marks in total so far.

d) is the only equivalence relation (1 more mark, but carry errors over).

(ii) d) is the only case where $\sim$ is an equivalence relation, and in this case we see $0 \sim x$ iff $-x \in \mathbb{Q}$ iff $x \in \mathbb{Q}$, so $cl(0)$ is just $\mathbb{Q}$. One more mark.

2)

(i) No and no, for two marks. The point is that in Q1(i), the relations a) and d) are both reflexive and symmetric, but a) is not transitive and d) is, so knowing $\sim$ is reflexive and symmetric doesn't tell you anything about whether it's transitive. Similarly, b) is reflexive and transitive but not symmetric, so knowing $\sim$ is reflexive and transitive doesn't tell you anything about whether it's symmetric.

(ii) Choose $s \in S$ (note that S is non-empty). Then $s \not\sim s$, so $\sim$ is not reflexive. On the other hand, if $a \sim b$ then certainly $b \sim a$, because $a \sim b$ is never true—if you're confused about this, then think about the statement "If I win the lottery, I will buy a new house"—this statement is true, even though I will never win the lottery. Similarly, transitivity is also true. Two more marks for this.

3) Messy stuff.

a) Say $\sim$ is an equivalence relation, and $\{T_i\}$ is the set of equivalence classes. We define a new relation $\star$ by $a \star b$ iff a, b are in the same T_i. We must check that $a \star b$ iff $a \sim b$. Well, firstly say $a \sim b$. Then $b \in cl(a)$ and $a \in cl(a)$, so a and b are in the same equivalence class, so $a \star b$. Conversely, if $a \star b$ then a, b are both in the same equivalence class, say, $cl(c)$, and hence $c \sim a$ and $c \sim b$, so by symmetry $a \sim c$ and by transitivity $a \sim b$. So indeed $a \star b$ iff $a \sim b$.

b) Given a partition $\{T_i\}$ of S, we define $\sim$ by $a \sim b$ iff a, b are in the same T_i and now we define a partition $\{U_j\}$ of S as being the equivalence classes of $\sim$. We want to prove that this partition $\{U_j\}$ is the same as $\{T_i\}$. So let's take one of the U_j. By definition it's $cl(a)$ for some $a \in S$. Now $\{T_i\}$ is a partition of S so $a \in T_i$ for some i. I claim that $U_j = T_i$ and this will suffice to prove that the partitions are the same, as U_j was an arbitrary element of the U partition. We know that $U_j = \{b \in S : a \sim b\}$ and by definition of $\sim$ we see that U_j is the set of $b \in S$ such that a, b are in the same set in the T partition. But $a \in T_i$ and hence U_j is just the set of b such that $b \in T_i$. Hence $U_j = T_i$ and we are home.

4*)

(i) We have to decide whether to let $1 \sim 1$, $1 \sim 2$, $2 \sim 1$, and $2 \sim 2$, so we have to make 4 yes-no decisions, so the total is 2^4. One mark.

(ii) This is a bit trickier. It's actually easier to count the relations which aren't symmetric. For symmetry to fail we must have $a, b \in S$ with $a \sim b$ and $b \not\sim a$. If $a = b$ then clearly this can't happen, so we must have $a \neq b$. So either $a = 1$ and $b = 2$ or $a = 2$ and $b = 1$. In the first case we have $1 \sim 2$ and $2 \not\sim 1$, and there are four such relations (because we can choose whether or not $1 \sim 1$ and whether or not $2 \sim 2$) and in the second case we have $2 \sim 1$ and $1 \not\sim 2$ so we get 4 more relations, giving a total of 8. Hence the $16 - 8 = 8$ relations left are symmetric. Three marks.

5) The proof is wrong because it might not be possible to choose b such that $a \sim b$—no such b might exist. Indeed, in Q2(ii) this situation occurs, and we see that Q2(ii) is an example of a symmetric transitive relation which isn't reflexive. So in fact Q2 implies that there is no redundancy amongst the axioms, in the sense that no two axioms imply the other.

6*) Four marks for this. The point is that if $a \in \mathbb{Z}$ then $a \sim a+8 \sim a+16$ and so by transitivity we have $a \sim a + 16$. But also $a + 1 \sim a + 6 \sim a + 11 \sim a + 16$ so by transitivity we have $a + 1 \sim a + 16$ and hence $a + 16 \sim a + 1$. So by transitivity again we have $a \sim a + 1$ and now by induction on n we easily see that $a \sim a + n$ for all $n \geq 1$. Reflexivity implies that $a \sim a$, and symmetry implies $a \sim a - n$ for all $n \geq 1$. Hence $a \sim b$ for all $a, b \in \mathbb{Z}$.