KMB, 29/11/01

M1F Foundations of analysis—Problem Sheet 9.

This is assessed coursework. Please hand in solutions to the starred questions (i.e., to questions 1, 2, 4, 6) at the end of the lecture on Monday 10th December.

1*)

(i) For each of the binary relations on the sets S below, decide (proof or counterexample) whether or not they are reflexive, symmetric, or transitive. Which ones are equivalence relations?

a) $S = \mathbb{R}$ and $x \sim y$ if $|x - y| < 1$.
b) $S = \mathbb{R}$ and $x \sim y$ if $x \leq y$.
c) $S = \mathbb{Z}$ and $x \sim y$ if $x + y$ is a multiple of 3.
d) $S = \mathbb{R}$ and $x \sim y$ if $x - y \in \mathbb{Q}$.

(ii) For the relations that did turn out to be equivalence relations in part (i), work out $cl(0)$, the equivalence class containing 0.

2*)

(i) If a relation on a set S is reflexive and symmetric, does it necessarily imply that the relation is transitive? If a relation on S is a reflexive and transitive, does it necessarily imply that it is symmetric? Hint: think about your answer to Q1.

(ii) Let S be a non-empty set, and define a relation $\sim$ on S by stating that $x \sim y$ is always false, for every x and y. This is the "empty relation". Is it reflexive? Symmetric? Transitive?

3) Let S be a set. In the course I proved

Proposition 7.1: if $\sim$ is an equivalence relation on S, then the equivalence classes form a partition of S.

Lemma 7.2: If $T_1, T_2, \ldots$ is a partition of S, then the relation on S defined by $a \sim b$ iff a and b are in the same T_i is an equivalence relation.

a) Prove that if $\sim$ is an equivalence relation on S, and $\{T_i\}$ is the partition one gets by invoking 7.1, then the equivalence relation one gets by invoking 7.2 again is equal to $\sim$

b) Prove that if $\{T_i\}$ is a partition of S, and $\sim$ is the equivalence relation one gets by invoking 7.2, then the partition one gets from $\sim$ by invoking 7.1 is $\{T_i\}$ again.

Hence there is a one-to-one correspondence between partitions of S and equivalence relations on S.

4*)

(i) How many binary relations are there on the set $S = \{1, 2\}$.

(ii) How many of these are symmetric?

5) Here is a proof that if a binary relation on a set S is symmetric and transitive, then it's also reflexive. Say $a \in S$. Choose b such that $a \sim b$. By symmetry, $b \sim a$. By transitivity, we have $a \sim b$ and $b \sim a$, so $a \sim a$.

But this proof is wrong—where is the mistake? Is it true that symmetric and transitive implies reflexive?

6*) Say $\sim$ is an equivalence relation on $\mathbb{Z}$, such that for all $a \in \mathbb{Z}$, we have $a \sim a + 8$ and $a \sim a + 5$. Show that $a \sim b$ for every $a, b \in \mathbb{Z}$.