KMB, 26/11/01

M1F Foundations of Analysis—Problem Sheet 8, hints and solutions.

1)

a) $n = 2r$ so $n^t = 2^t r^t$ is clearly a multiple of 2^t.

b1) If x is even, then $y^5 = x^8 - 4$ is even. Now if y is odd then $y^5 = y.y.y.y.y$ is odd. Hence y must also be even. By part (a) we see that $2^5 = 32$ divides y^5, and 2^8 divides x^8, so in particular 32 also divides x^8. We deduce that 32 divides $x^8 - y^5 = 4$, which is a contradiction. So there is no solution with x even.

b2) Rearrange the equation and deduce that $y^5 = x^8 - 4 = (x^4 - 2)(x^4 + 2)$. Now let's wonder whether those two terms on the right hand side are coprime.

If d is a positive integer dividing $x^4 - 2$ and $x^4 + 2$, then d divides their difference, which is 4. In particular, $d = 1$ or 2 or 4. But x is assumed odd, so $x^4 - 2$ is odd. Hence d cannot be 2 or 4, and must be 1. So $x^4 - 2$ and $x^4 + 2$ are coprime.

Now just before we go ahead and apply the theorem that says that $x^4 - 2$ and $x^4 + 2$ are 5th powers, we should check to see that the conditions of the theorem hold. And actually we have to check one thing: maybe $x^4 - 2$ is not positive. But this can only occur when $x = \pm 1$ and we see easily that if $x = \pm 1$ then y is not an integer.

Hence $|x| \geq 3$, and $x^4 - 2$ and $x^4 + 2$ are both positive. Hence y is positive and now we can deduce that $x^4 + 2$ and $x^4 - 2$ must both be 5th powers.

But $x^4 + 2$ and $x^4 - 2$ differ by 4, and if we look at the list of 5th powers, it goes $\ldots - 243, -32, -1, 0, 1, 32, 243 \ldots$ with the differences between consecutive 5th powers getting very big. Hence no 5th powers differ by 4, and there are no solutions with x odd.

2)

a) $a = 6$, $b = 10$, $c = 15$ will do.

b) $a = 2.3.5$, $b = 2.7.11$, $c = 3.7.13$ and $d = 5.11.13$ solves this tricky problem. The idea is that for each pair of elements of S, just choose a prime that will divide exactly these two elements and no other.

3)

(i) Some experimentation shows us that $3^6 \equiv 1 \bmod 7$ and hence $3^{1998} = (3^6)^{333} \equiv 1^{333} \equiv 1 \bmod 7$. Hence $3^{2001} = 3^{1998}.3^3 \equiv 3^3 \equiv 27 \equiv 6 \bmod 7$.

(ii) We see that (working mod 17 all the time), $7^1 \equiv 7$, $7^2 = 49 \equiv -2$, $7^4 \equiv (-2)^2 \equiv 4$, $7^8 \equiv 4^2 \equiv -1$ and $7^{16} \equiv (-1)^2 \equiv 1$. So $7^{32} \equiv 1$ and $7^{64} \equiv 1$ as well, and $7^{80} \equiv 7^{16+64} \equiv 1 \bmod 17$.

4)

(i) Every integer n is congruent to either $0, 1, 2, 3, 4, 5, 6$ or 7 mod 8, and so it suffices to check for these cases. We see that $0^2 \equiv 4^2 \equiv 0 \bmod 8$, $1^2 \equiv 3^2 \equiv 5^2 \equiv 7^2 \equiv 1 \bmod 8$, and $2^2 \equiv 6^2 \equiv 4 \bmod 8$. So we are done.

(ii) Again it suffices to check for $n = 0, 1, 2, 3, 4$. We see $0^5 = 0$ and $1^5 = 1$. Now $2^5 = 32 \equiv 2 \bmod 5$, $3^5 \equiv (-2)^5 \equiv -2^5 \equiv -2 \equiv 3 \bmod 5$, and $4^5 \equiv (-1)^5 \equiv -1 \equiv 4 \bmod 5$, so again we are done.

5) n is congruent to the sum of its digits mod 9, and the sum of the digits is $9+(7+2)+(6+3)+(5+4)+8 \equiv 8 \bmod 9$. The remainder when n is divided by 11 will be congruent to $2-3+4-5+6-7+8-9 = (2-3)+(4-5)+(6-7)+(8-9) = -4 \equiv 7 \bmod 11$. Finally, 32 is a multiple of 4 and hence so is n.

For the last part, note that we know $n = 9t + 8$ for some t, and hence the remainder when we divide n by 3 is the same as the remainder when we divide 8 by 3, which is 2.

6)

(a) The fact that 13 divides 1001 means that $10^3 \equiv -1 \bmod 13$ and so we are in some kind of generalisation of the rule of 11. The way it works: split n up into chunks of size 3 and then add and subtract alternate terms and the result is congruent to n mod 1001 and hence mod 13 (and also mod 7 and mod 11). For example, if $n = 6,005,004,003,002,001$ then we see that n mod 13 will be $1-2+3-4+5-6 \equiv -3 \equiv 10 \bmod 13$.

b) We see that $10^3 \equiv 1 \bmod 37$ and hence $10^{3t} \equiv 1 \bmod 37$ for all $t \geq 1$. Hence we can generalise the rule of 9, and we see that we should break n up into blocks of three, add them up, and the result will be congruent to n mod 999 and hence mod 37. For example, $6005004003002001 \equiv 6+5+4+3+2+1 \equiv 21 \bmod 37$.

7) A hint: Consider the product 2^n5^n.