M1F Foundations of analysis—Problem Sheet 8.

1)

a) Show that if n is an even integer, and $t \geq 1$ is an integer, then n^t is a multiple of 2^t.

b) Show that if x and y are integers, then $x^8 - y^5$ can never equal 4, in the following way: Assume that there is a solution in integers x and y.

Case 1: If x is even, deduce that y is even and now use (a) to deduce a contradiction.

Case 2: If x is odd, re-arrange and use a clever factorisation to show that there is also a contradiction in this case.

2)

a) Find three positive integers a, b, c, such that there is no prime that divides all three of a, b and c, but $\text{hcf}(a,b)$, $\text{hcf}(b,c)$ and $\text{hcf}(c,a)$ are all greater than one.

b) Can you find a set $S = \{a,b,c,d\}$ of four distinct positive integers, such that if x and y are any two distinct elements of S, then $\text{hcf}(x,y) > 1$, but also, if r, s, t are any three distinct elements of S, then no prime divides all three of r, s, t? Not the best explained question of all time, but I'm sure you get my drift.

3) (i) Find an integer t between 0 and 6, such that $3^{2001} \equiv t \bmod 7$. Hint: work out 3^n mod 7 for a few small values of n and take it from there.

(ii) Use the method of repeated squares to find the remainder when 7^{80} is divided by 17.

4) (i) For $n \in \mathbb{Z}$, orove that n^2 is congruent either to 0, 1 or 4 mod 8.

(ii) For $n \in \mathbb{Z}$, prove that n^5 is congruent to n mod 5.

5) Set $n = 98765432$. Using the rules of 9, 11 and 4, work out the remainders when n is divided by 9, 11, and 4. What is the remainder when n is divided by 3? (hint: you can read off the answer from what you already have, if you think).

6) a) Use the fact that 13 divides 1001 to work out a rule of 13. Using this rule, and again not using a calculator, work out the remainder when 6005004003002001 is divided by 13.

b) $999 = 27 \times 37$. Using this fact, work out a rule of 37 and hence work out the remainder when 6005004003002001 is divided by 37.

$7^{\dagger}$) There is an integer $n \in \mathbb{N}$ such that 2^n and 5^n both begin with the same three digits. Without using a calculator or computer, work out what these three digits are.