KMB, 19/11/01

# M1F Foundations of Analysis—Problem Sheet 7, hints and solutions.

1*)
(i)

$$\begin{aligned} 37 &= 1.21 + 16, \\ 21 &= 1.16 + 5, \\ 16 &= 3.5 + 1, \\ 5 &= 5.1 + 0. \end{aligned}$$

So we terminate, and deduce that the highest common factor is $d = 1$ (one mark). Now to find $\lambda$ and $\mu$ we see

$$\begin{aligned} 16 &= 37 - 21, \\ 5 &= 21 - 16 = 21 - (37 - 21) = 2.21 - 37, \\ 1 &= 16 - 3.5 = (37 - 21) - 3(2.21 - 37) = 4.37 - 7.21. \end{aligned}$$

We finally check (this is not mathematically essential, but it's very wise!) and note that $4.37 = 148$ and $7.21 = 147$ so indeed we have not made a mistake and our answer is $\lambda = -7$ and $\mu = 4$. Two more marks for this, but lose a mark if you get them the wrong way around, or if you miss a sign or two. Note that there are other *correct* choices for $\lambda$ and $\mu$, for example $\lambda = 30$ and $\mu = -17$, and of course you won't lose a mark if you write down one of these other correct answers.

(ii) Working all on one line to save space, $437 = 247 + 190$, $247 = 190 + 57$, $190 = 3.57 + 19$ and $57 = 3.19 + 0$ so $d = 19$ (one mark) and $190 = 437 - 247$, $57 = 247-(437-247) = 2.247-437$, $19 = 190-3.57 = (437-247)-3.(2.247-437) = 4.437 - 7.247$ (now check this!) and so $\lambda = -7$ and $\mu = 4$ (or any of the other infinitely many correct solutions) (two marks for getting a correct $\lambda$ and $\mu$, lose one if you slip up with a sign or the order).

2)

(i) By Proposition 6.4 there are integers $\lambda$ and $\mu$ such that $\lambda a + \mu b = 1$. So one can set $s = n\lambda$ and $t = n\mu$; then $sa + tb = n(\lambda a + \mu b) = n$ for an easy two marks.

(ii) Certainly $\lambda = -2$ and $\mu = 1$ gives us $3\lambda + 7\mu = 1$, but unfortunately this $\lambda$ is too small. But by the trick in lectures, let's add, say, $7,000,000$ to $\lambda$ and subtract $3,000,000$ from $\mu$; this keeps $3\lambda + 7\mu$ constant at 1, and we get $\lambda = 6,999,998$ and $\mu = -2,999,999$ as a solution that satisfies the inequality. A measly two marks (well, I told you how to do it in lectures!)

3†) As I said in lectures, I forgot to write $\text{hcf}(a, b) = 1$ on the question sheet.

If $a = 1$ or $b = 1$ then clearly one can buy any number of McNuggets. So the interesting case is when $a, b > 1$. Some messing around with specific examples will convince you now that the largest number one cannot buy is $t = ab - a - b$ (note that $t + 1 = (a - 1)(b - 1) \geq 1$ so $t \geq 0$). To prove this, one needs to be able firstly to show that one cannot buy $t$ McNuggets, and secondly that one can buy any integer that is greater than $t$.

I'll firstly explain how to buy $n > t$ McNuggets. By Q2(i) for any integer $n > t$ we can find integers $\lambda$ and $\mu$ such that $\lambda a + \mu b = n$. By the trick in Q2(ii) we can assume $\lambda > 0$. Divide $\lambda$ by $b$ and let $r$ be the remainder, with $0 \leq r < b$. We know that $n - \lambda a$ is a multiple of $b$, and so $n - ra$ is also a multiple of $b$. If $s = (n - ra)/b$ then $s$ is an integer. Furthermore, $s = (n - ra)/b > (t - ra)/b \geq (t - (b-1)a)/b \geq -1$ and hence $s > -1$, so $s \geq 0$ and $n = ra + sb$ McNuggets can be bought.

Next I'll explain why we can't buy $t$. Say $ab - a - b = ra + sb$ with $r, s \geq 0$. We then see that $(s+1)b = a(b-1-r)$ and hence $a$ divides $(s+1)b$. But $\mathrm{hcf}(a, b) = 1$ and so by Corollary 6.5, $a$ divides $s+1$ and because $s+1 > 0$ we have $s+1 \geq a$. So $s \geq a - 1$. Hence $t = ra + sb \geq ra + ab - b \geq ab - b > ab - a - b = t$, a contradiction.

4*) We must find the smallest positive integer $N$ such that $N$ is a multiple of both 28 and 35. We could either invoke the uniqueness of prime factorization theorem, and deduce that $N$ must be a multiple of 4, 5, and 7, and hence $N \geq 140$, and hence $N = 140$ (which clearly works) will be the minimum, or alternatively we could try and solve $28a = 35b$ by more low-level methods: we see that $28a = 35b \implies 4a = 5b$, so $4a$ is a multiple of 5, and by Corollary 6.5 we deduce that 5 divides $a$, so $a \geq 5$, and so the solution $a = 5$ and $b = 4$ is the smallest, again giving the answer as 140 minutes past midday, that is, 2:20pm, or 1420 if you come from a country where they have the sense to use the 24 hour clock. One final method would just be trial and error, perhaps I should have made the numbers larger to make this approach harder. Four marks for all this, because I didn't really do anything like it in the lectures.

5*)

(i) It is true. The simple proof is that if $g$ is any integer dividing $a$ and $b$ then by Lemma 6.2 $g$ divides $sa + tb$ too, so $g$ divides 1, so $g = \pm 1$. So $\mathrm{hcf}(a, b) = 1$. Three marks.

(ii) It is not true—for example $a = b = s = t = 1$. Three generous marks.

7) If $\mathrm{hcf}(a, b) = \mathrm{hcf}(a, c) = 1$ then there are integers $\alpha$, $\beta$, $\gamma$, $\delta$ such that $\alpha a + \beta b = 1$ and $\gamma a + \delta c = 1$. Multiplying these equations gives $(\alpha\gamma a + \beta\gamma b + \alpha\delta c)a + (\beta\delta)(bc) = 1$ and so $sa + tbc = 1$ for some integers $s$ and $t$, so by Q5(i) we have $\mathrm{hcf}(a, bc) = 1$.