M1F Foundations of analysis—Problem Sheet 7.

This is assessed coursework. Please hand in solutions to the starred questions (i.e., to questions 1, 2, 4, 5) at the end of the lecture on Monday 26th November.

1*) For each of the following pairs of positive integers a, b, find the highest common factor $d = \mathrm{hcf}(a, b)$ and also integers λ and μ such that $\lambda a + \mu b = d$.

(i) $a = 21$, $b = 37$.

(ii) $a = 247$, $b = 437$.

2*)

(i) Say a and b are positive integers and $\mathrm{hcf}(a, b) = 1$. Prove that for any $n \in \mathbb{Z}$ there exist integers s and t (of course, depending on n) such that $n = sa + tb$.

(ii) The highest common factor of 3 and 7 is 1. Find integers λ and μ such that $3\lambda + 7\mu = 1$ and such that $\lambda > 10^6$.

3†) Generalised McNuggets. Prove that if $a, b \in \mathbb{N}$ have highest common factor 1, and there is a branch of McDonalds where Chicken McNuggets come in either a pack of size a or a pack of size b, then for all sufficiently large integers N it is possible to buy exactly N McNuggets. What is the largest number of McNuggets that one cannot buy in this case?

4*) I have a broken clock which only ticks once every 28 minutes. My friend has a broken clock which only ticks once every 35 minutes. Both clocks ticked at midday today. When will they next both tick again?

5*) Proof or counterexample required. Let a, b be positive integers.

(i) If there are integers s and t such that $sa + tb = 1$, then is it true that $\mathrm{hcf}(a, b) = 1$?

(ii) If there are integers s and t such that $sa + tb = 2$, then is it true that $\mathrm{hcf}(a, b) = 2$?

6) Without assuming the Fundamental Theorem of Arithmetic, prove that if $a, b, c \in \mathbb{N}$ and $\mathrm{hcf}(a, b) = \mathrm{hcf}(a, c) = 1$ then $\mathrm{hcf}(a, bc) = 1$. Hint: use 5(i) and Proposition 6.4.