KMB, 12/11/01

M1F Foundations of Analysis—Problem Sheet 6, Hints and Solutions.

1) a) LUB $= -1$, GLB $= -3$ (not the other way round!)
b) LUB $= -1$, no GLB.
c) LUB $= \sqrt{2}$, GLB $= -\sqrt{2}$.
d) No LUB or GLB.
e) LUB $= \sqrt{2}$, GLB $= -\sqrt{2}$.

2) (Of course there are many many answers to these questions.)
a) $\{1, 1.4, 1.41, 1.414, 1.4142, \ldots\}$, the set of "decimal approximations" to $\sqrt{2}$, has LUB equal to $\sqrt{2}$ (as long as you remember to always round down).
b) $\{-\sqrt{2}, -\sqrt{2}/2, -\sqrt{2}/3, -\sqrt{2}/4, \ldots\}$ is a set of negative irrationals, most of which are very small, and the LUB is 0.
c) $\{-1, -1.4, -1.41, -1.414, -1.4142, \ldots\}$ works.

3)
a) If x is a lower bound for B, then for all a in A, we have $a \in B$, so $x \leq a$. Hence x is a lower bound for A.
b) We know that y is a lower bound for B. Hence by (a), y is a lower bound for A. But x is the greatest lower bound for A. So $x \geq y$.

4)
a) Let $y = x + c$. We have to show that y is a GLB for T. First let's check that y is a lower bound! If $t \in T$, then $t - c \in S$, so $x \leq t - c$ (as x is a lower bound for S), so $y = x + c \leq t$. Hence y is a lower bound.

Now let's check that y is at least as big as any other lower bound. To do this, let z be any lower bound for T. By a similar argument to the above, one can check that $z - c$ is a lower bound for S. But x is the GLB of S, so $z - c \leq x$. So $z \leq x + c = y$.

Hence, by definition of GLB, y is the greatest lower bound for S.

b) Again, one has to check both the parts of the definition for a LUB. Firstly, $-x$ is an upper bound, because if $-s \in T$, then $s \in S$, so $x \leq s$, so $-x \geq -s$.

And secondly, if u is an upper bound for T, then for all t in T we have $u \geq t$, so for all s in S we have $-u \leq s$. Hence $-u$ is a lower bound for S, and so $-u \leq x$. Hence $u \geq -x$ and so $-x$ is the least upper bound for T.

5) Certainly all of the T_i are non-empty, because $x_i \in T_i$. Hence it suffices to show that all of the T_n have lower bounds. But $T_n \subseteq T_1$, so by Q4(a), any lower bound for T_1 is a lower bound for all the T_n. So they all have GLBs.

Next note that $T_{n+1} \subseteq T_n$ for all $n \geq 1$, so by Q4(b), $b_{n+1} \geq b_n$ for all $n \geq 1$.
a) $b_n = n$ and so the set of all b_i has no upper bound.
b) $b_n = 0$ for all n and so the upper bound is 0.
c) $b_n = 1$ for all n and so the upper bound is 1.
d) $b_n = 1$ for $n \leq 100$ and $b_n = 2$ for $n > 100$, and so $\{b_1, b_2, b_3, \ldots\} = \{1, 2\}$ and the upper bound is 2.

Remark: In general, if the set T_1 is bounded above and below, then the b_i will always be bounded above, and so their LUB will exist, so the liminf always exists. Guess what a limsup is and check that this exists too.