M1F Foundations of analysis—Problem Sheet 6.

1) Write down (no explanation necessary) the least upper bounds and greatest lower bounds of the sets below, or say "none" if they don't exist.

a) $\{-1, -2, -3\}$ b) $\{-1, -2, -3, \ldots\}$
c) $\{x \in \mathbb{Q} : x^2 < 2\}$ d) $\{x \in \mathbb{Q} : x^2 > 2\}$
e) $\{x \in \mathbb{R} : x \text{ is irrational, and } x^2 < 2\}$

2) Give examples of:
a) A set of rationals with a least upper bound that is irrational.
b) A set of irrationals with a least upper bound that is rational.
c) A set of rationals with a rational upper bound but an irrational lower bound.

3) Write down proofs of the following. Both proofs should be only a few lines long.
a) If $A \subseteq B$ are sets of reals, then a lower bound for B is also a lower bound for A.
b) If $A \subseteq B$ are sets of reals, and the greatest lower bound of A is x, and the greatest lower bound of B is y, then $y \leq x$.

4) Give proofs of the following two steps that I skipped during the proof of the least upper bound theorem:
a) Let c be a real number. Let S be a set of reals, with greatest lower bound x. Let T be the set that you get by "shifting everything in S up by c". That is, $T = \{x \in \mathbb{R} : x - c \in S\}$. Prove that $x + c$ is a greatest lower bound for T. [Don't just say "it's obvious"—but actually check that $x + c$ satisfies the definition of a greatest lower bound for T.]
b) Let S be a set of reals, and let T denote the set "$-S$". More formally, set $T = \{x \in \mathbb{R} : -x \in S\}$. Prove that if S has a greatest lower bound, x, then $-x$ is a least upper bound for T.

5) Let x_1, x_2, $x_3, \ldots$ be a sequence of real numbers. For any integer $n \geq 1$, define the set T_n to be the set $\{x_n, x_{n+1}, x_{n+2}, \ldots\}$. So for example,

$$T_1 = \{x_1, x_2, x_3, x_4, \ldots\}$$

and

$$T_2 = \{x_2, x_3, x_4, \ldots\},$$

and so on.
Assume that T_1 has a lower bound. Deduce that T_n has a greatest lower bound for all $n \geq 1$. Call this lower bound b_n. Show that $b_1 \leq b_2 \leq b_3 \ldots$.
Sometimes the set $\{b_1, b_2, b_3 \ldots\}$ has an upper bound. For the sequences (x_m) below, work out b_n and also the upper bound B of the set $\{b_1, b_2, b_3 \ldots\}$, when it exists. The number B is called the *liminf* of the sequence $x_1, x_2, \ldots$.
a) $x_1 = 1$, $x_2 = 2$, $x_3 = 3 \ldots$, and in general $x_n = n$.
b) $x_1 = 1$, $x_2 = 1/2$, $x_3 = 1/3 \ldots$, and in general $x_n = 1/n$.
c) $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 2$, $x_5 = 1$ and so on, alternating between 1 and 2.
d) $x_1 = x_2 = \ldots = x_{100} = 1$ and $x_n = 2$ for all $n > 100$.