M1F Foundations of Analysis—Problem Sheet 5, hints and solutions.

My experience is that the most common mistake people make in these induction questions is that they let $P(n)$ be a function, not a statement, and end up trying to prove the number "$n(n+1)(2n+1)/6$" by induction. So lose a mark for each time this happens, up to a maximum of three marks.

1*)

(i) Let $P(n)$ denote the statement that $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$. We have to prove $P(n)$ for all $n \in \mathbb{N}$ and we will of course do this by induction. Firstly $P(1)$ says that $1 = 1$, so it's true. Next, if we know $P(n)$, then $\sum_{i=1}^{n+1} i^2 = n(n+1)(2n+1)/6 + (n+1)^2 = (n+1)(n^2/3 + n/6 + n + 1) = \frac{1}{6}(n+1)(2n^2+7n+6) = \frac{1}{6}(n+1)(n+2)(2n+3)$ which is $P(n+1)$, so $P(n) \Rightarrow P(n+1)$ and we are home. Two marks.

(ii) We use slightly modified induction to prove the statement $P(n)$:"$5^n > 4^n + 3^n + 2^n$" by induction on $n \geq 3$. We check that $P(3)$ is true: indeed $P(3)$ says that $125 > 64 + 27 + 8 = 99$. For general $n \geq 3$ we see that $P(n)$ implies $5^{n+1} = 5.5^n > 5.4^n + 5.3^n + 5.2^n > 4.4^n + 3.3^n + 2.2^n = 4^{n+1} + 3^{n+1} + 2^{n+1}$ and this is $P(n+1)$, so we are done by induction. Three marks.

2*)

(i) If $P(n)$ is the statement that $n + (n+1) + (n+2) + (n+3)$ is always a multiple of 4, plus two, then in fact, now I'm typing this up, I realise that $P(n)$ is true and you don't even need induction. But let's prove it by induction anyway: $P(1)$ says that 6 leaves remainder 2 when divided by 4, which is true, and for general n, if we know $P(n)$ then we know that $n+(n+1)+(n+2)+(n+3) = 4t+2$ and hence $(n+1) + (n+2) + (n+3) + (n+4) = 4t + 6 = 4(t+1) + 2$ is also a multiple of 4, plus two. Blame house-buying stress for this question! Two marks.

(ii) Let $P(n)$ denote the statement that $10^n - 2^{2n}$ is a multiple of 6. We need to prove $P(n)$ for all $n \geq 0$, so we use slightly modified induction. We check that $P(0)$ is true because $10^0 - 2^0 = 0$. For general $n \geq 0$, if $P(n)$ is true, then $10^n - 2^{2n} = 6t$ for some integer t, and hence $10^{n+1} - 10.2^{2n} = 60t$, so $10^{n+1} - 2^{2n+2} = 10^{n+1} - 4.2^{2n} = 60t + 6.2^{2n}$ is clearly a multiple of 6, and so we are done by induction. Three marks, lose one for not dealing with $n = 0$.

3*) Looking at remainders when we divide the terms of the sequence by 3, we see that it seems to go $1, 0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, 0, 2, 2, \ldots$ and that the pattern $1, 0, 1, 1, 2, 0, 2, 2$ seems to be repeating. To prove this we should use induction, and we capture the entire statement of what we want to prove in the following way: for $n \geq 0$ let $P(n)$ denote the following statement:

$P(n)$:"The remainder when you divide the eight numbers $L_{8n+1}, L_{8n+2}, L_{8n+3}, \ldots, L_{8n+8}$ by 3 are $1, 0, 1, 1, 2, 0, 2, 2$ respectively".

Now $P(0)$ is easily checked by hand, and $P(n)$ implies $P(n+1)$ easily by induction, because if we know the remainder when we've divided L_{8n+7} and L_{8n+8} by 3 then from this we can work out the next 8 remainders term by term using the definition of L_n. In particular, because $2001 = 250.8 + 1$ we see that L_{2001} will leave remainder 1 when you divide it by 3.

Alternatively one could prove by strong induction the statement $P(n)$: L_n and L_{n+8} leave the same remainder when divided by 3. This in fact is rather neater. You check by hand for $n = 1$ and $n = 2$, and then for $n \geq 2$, if you know $P(n-1)$ and $P(n)$ then you know that $L_{n-9} - L_{n-1}$ and $L_{n-8} - L_n$ both are multiples of 3, and hence so is their sum, which is $L_{n-7} - L_{n+1}$, and so $P(n+1)$ is true. In retrospect I like this argument better.

Six well-earned marks for a complete and correct presentation of this proof, and partial attempts should get two or so marks, but more if one of the key ideas (for example, a statement $P(n)$ that is provable by induction) is there. Note that L_{2001} has 419 digits, this sequence is growing pretty fast.

Note also that it seems impossible to prove the statement that L_{8n+1} leaves remainder 1 when divided by 3, alone, by induction on n. One has to prove something stronger! It's a bit strange at first that it's easier to prove something harder, but this happens sometimes with induction.

4*) The interior angles of a pentagon are 108 degrees, so four or more of them cannot meet at one vertex. So all vertices must have three edges coming from them. If there are f faces in the polyhedron then there will be $5f/2$ edges and $5f/3$ vertices, by an argument similar to the one I gave for the octohedron in the course: the f pentagons have $5f$ vertices and $5f$ edges, but each edge is counted twice and each vertex three times. By the $F + V = E + 2$ formula we have $f + 5f/3 = 5f/2 + 2$ and solving this gives $f/6 = 2$ and hence $f = 12$. An easyish four marks. No marks for nonsense like "it's obvious that the only way to fit them together is into a dodecahedron, so we are done"!

5)

(i) If $n = 0$ we get one region, but if $n \geq 1$ then we get $2n$ regions, and one can prove this by induction on $n \geq 1$: if $P(n)$ is the statement that n lines passing through 1 point cut the plane into $2n$ regions, then assume $P(n)$ and imagine n lines going through one point. Draw another line in. This breaks two regions each up into two pieces, and hence adds two new regions. Now $2n + 2 = 2(n + 1)$ so $P(n)$ implies $P(n + 1)$ and we are home.

(ii) Trying experiments we see that for $n = 0, 1, 2, 3, 4, \ldots$ the answer comes out as $1, 2, 4, 7, 11, \ldots$, and in trying to spot the pattern we look at consecutive differences and notice that they are $1, 2, 3, 4, \ldots$. Recalling the formula for $1 + 2 + \ldots + n$, we conjecture that the number of regions we get with n lines is $1 + n(n + 1)/2$. The proof is straightforward induction: the point is that if you have n lines in the plane, and you draw an $n + 1$st, then this line will hit all n of the other lines, and these n points of intersection divide this last line into $n + 1$ sub-lines, each of which is chopping one old region into 2. So all we have to do is to check that $(1 + n(n + 1)/2) + (n + 1) = 1 + (n + 1)(n + 2)/2$, which is easy algebra.

6) Some playing around gives that it seems to be impossible to buy 43 Chicken McNuggets, but that we can do 44 ($= 20 + 6 + 6 + 6 + 6$), and 45 ($= 9 + 9 + 9 + 9 + 9$) and 46 ($= 20 + 20 + 6$) and 47 ($= 20 + 6 + 6 + 6 + 9$) and 48 ($= 6 + 6 + 6 + 6 + 6 + 6$) and 49 ($= 20 + 20 + 9$) (one mark).

Now we have to do two things: firstly, prove that we can't get 43, and secondly, that we can get everything higher than 43. To show that everything higher than 43 is buyable, we will do by the principle of strong induction. Let $P(n)$ be the statement that it was possible to buy n McNuggets in those bygone years. Then we just showed above that $P(44)$ up to $P(49)$ are all true. I claim that $P(n)$ is true for all $n \geq 44$ by strong induction.

We have already checked $P(44)$, so we can start. If $k > 44$, we have to check that $P(44)$, $P(45), \ldots, P(k-1)$ imply $P(k)$. Well, if $45 \leq k \leq 50$ then we have already proved $P(k)$ without assuming anything at all! So certainly we are OK if $k < 50$. For if $k \geq 50$, we have $k - 6 \geq 44$, and so we are allowed to assume $P(k-6)$, and can deduce $P(k)$ [because if we can buy $k-6$ McNuggets, we then just buy another box of 6 on top of that, and we have k McNuggets.] So by strong induction, we can buy n McNuggets for any $n > 43$.

[Alternatively, one can use normal induction—if one chooses a much cleverer $P(n)$. For example, if $P(n)$ is the statement "It is possible to buy $6n + 44$, and $6n + 45$, and $6n + 46$, and $6n + 47$, and $6n + 48$, and $6n + 49$ McNuggets", then $P(0)$ can be checked, and for $k \geq 0$ it's easy to check that $P(k)$ implies $P(k+1)$, so all $P(n)$ are true for $n \geq 0$.]

Now what about $n = 43$? Well, 43 is not a multiple of 3. So if we want to buy 43, we can't just use portions of 6 and 9, we're going to have to buy a pack of 20. Once we've done this, we still have to buy 23, which still is not a multiple of 3, and hence we have to buy another 20. Now we have to buy three more, and there's no way we can just buy three. So 43 McNuggets cannot be bought (two marks for this part).

Hence 43 is the largest number of McNuggets that you can't buy.