KMB, 01/11/01

M1F Foundations of analysis—Problem Sheet 5.

This is assessed coursework. Please hand in solutions to the starred questions (i.e., to questions 1, 2, 3, 4) at the end of the lecture on Monday 12th November.

1*) Prove the following statements by induction:
(i) If $n \in \mathbb{N}$, then $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.
(ii) If $n \geq 3$ is an integer, then $5^n > 4^n + 3^n + 2^n$.

2*) More of the same: Prove the following statements by induction:
(i) The sum of four consecutive positive integers always leaves remainder 2 when divided by 4.
(ii) For all integers $n \geq 0$, the number $10^n - 2^{2n}$ is a multiple of 6.

3*) The *Lucas sequence* is defined as follows: $L_1 = 1$, $L_2 = 3$ and, for $n \geq 3$, $L_n = L_{n-1} + L_{n-2}$ (like the Fibonacci sequence). So the sequence begins $1, 3, 4, 7, 11, 18, \ldots$. What is the remainder when L_{2001} is divided by 3? Proof required, of course!

4*) Prove that any convex polyhedron all of whose faces are pentagons must have precisely 12 faces.

5)
(i) If we have n (distinct) straight lines in the plane, all passing through a single point, how many regions does this chop the plane into?
(ii) If we have n distinct straight lines in the plane, and no two are parallel, and there is no point in the plane that three of the lines pass through, then how many regions does this chop the plane into?

6) When I was a student, Chicken McNuggets used to be sold in boxes of 6 or 9, and also in a "family pack" containing 20. A friend of mine once asked me what was the largest number $n \in \mathbb{N}$ of Chicken McNuggets that it was impossible to buy. For example, it was possible to buy 12 Chicken McNuggets $(= 6+6)$ or 15 $(= 6+9)$ but it was not possible to buy 13 because you couldn't buy single McNuggets or anything like that. So what was the answer to my friend's question?