M1F Foundations of Analysis—Problem Sheet 4, hints and solutions.

1)

(a) $(1+i) = \sqrt{2}\big(\cos(\theta) + i\sin(\theta)\big)$, where $\theta = \frac{\pi}{4}$. So in polar form, $1+i = (\sqrt{2}, \pi/4)$ and hence $(1+i)^{100} = (\sqrt{2}^{100}, 100\pi/4) = (2^{50}, 25\pi) = -2^{50}$.

(b) The expression is just the real part of $(1+i)^{100}$ and hence is equal to -2^{50}.

(c) One can prove that for all $n \geq 0$ we have $\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k} = (-4)^n$, by induction (repeatedly apply $\binom{a}{b} + \binom{a+1}{b} = \binom{a+1}{b+1}$), and the result follows if one sets $n = 25$. A bright combinatorics PhD student showed me a direct proof not involving complex numbers or induction, involving counting the number of ways you can colour in n squares from a $2 \times n$ grid of squares, I'll tell anyone who is interested but it's a bit long to type in.

2) In polar coordinates, we have $1+i = (\sqrt{2}, \pi/4)$ and $\sqrt{3}+i = (2, \pi/6)$. Hence $\frac{1+i}{\sqrt{3}+i} = (1/\sqrt{2}, \pi/12)$. But working algebraically we see that $\frac{1+i}{\sqrt{3}+i} = \frac{(1+i)(\sqrt{3}-i)}{4} = \frac{\sqrt{3}+1+i(\sqrt{3}-1)}{4}$. Now bashing it all out and equating real parts, we deduce that $\frac{1}{\sqrt{2}}\cos(\frac{\pi}{12}) = \frac{\sqrt{3}+1}{4}$, and the result follows easily.

The value is irrational—if it were rational then $\sqrt{6}+\sqrt{2}$ would be rational, and hence $\sqrt{6} = \sqrt{2} + \frac{a}{b}$; squaring, we deduce that an irrational number is rational, a contradiction.

3)

(a) The 10 roots are $\cos(\theta) + i\sin(\theta)$ for $\theta = \pi/20 + 2s\pi/10$, $0 \leq s < 10$. The root closest to i will correspond to the θ nearest to $\pi/2$, which will correspond to $s = 4$.

(b) The three cube roots 1, ω and ω^2 of 1 are on the vertices of an equilateral triangle—this can be checked explicitly by noticing, for example, that all three sides have length $\sqrt{3}$. For a general complex number $z \neq 0$, if ζ is one cube root of z, then the other two are $\zeta\omega$ and $\zeta\omega^2$, hence the sides of the triangle they form all have length $\sqrt{3}|\zeta|$, and again the triangle is equilteral.

4)

a) $\zeta \neq 1$ because it has non-zero imaginary part. Hence $1+\zeta+\zeta^2+\zeta^3+\zeta^4 = (\zeta^5-1)/(\zeta-1) = 0/(\zeta-1) = 0$.

b) Note that $\alpha+\beta = -1$ by part a), and $\alpha\beta = \zeta^3+\zeta^4+\zeta^6+\zeta^7 = \zeta^3+\zeta^4+\zeta+\zeta^2 = -1$. So $X^2+X-1 = (X-\alpha)(X-\beta)$ (expand out) as required.

c) The roots of X^2+X-1 are $X = \frac{-1\pm\sqrt{5}}{2}$. Moreover, $\alpha = 2\cos(2\pi/5)$ and $\beta = 2\cos(4\pi/5)$. Which is which? Well, $2\cos(2\pi/5)$ is definitely positive, as $0 < 2\pi/5 < \pi/2$, so $2\cos(2\pi/5) = \frac{\sqrt{5}-1}{2}$ and now we are home.

5) The answer to this is shorter than the question.

a) Note that $(r-s)^2 = (r+s)^2 - 4rs$, so if we know $r+s$ and rs then we know $(r-s)^2$, so we know $r-s$ up to sign, and now $r = \big((r+s)+(r-s)\big)/2$ and $s = \big((r+s)-(r-s)\big)/2$. Alternatively note that r and s are the roots of the polynomial $X^2-(r+s)X+rs$, this is the trick we used in 4b in fact.

b) The roots of $ax^3 + bx^2 + cx + d$ are clearly the same as the roots of $x^3 + (b/a)x^2 + (c/a)x + (d/a)$. Moreover, if we now set $y = (x + b/3a)$ then messy multiplying out shows that $y^3 + ey + f = 0$ for suitably-chosen e and f. Hence if we have a general formula for y, then we have a general formula for $x = y - b/3a$.

c) Easy, just multiply out.

d) Again it's very straightforward. We see that $M^3 - N^3 = \pm\sqrt{B^2 + 4A^3/27}$ and if we choose a sign then we get $M^3 = (-B + \sqrt{B^2 + 4A^3/27})/2$ and $N^3 = (-B - \sqrt{B^2 + 4A^3/27})/2$. Hence M is the cube root of that mess, $N = -A/3M$, and $x = M + N$ is a root of the equation. In fact we do get all three roots this way. Firstly note that the choice of square root doesn't matter, because if we switch M and N around then $M + N$ is still the same. Next note that we have three choices for the cube root of M^3 and this yields the three roots of the cubic, which are easily checked to be distinct in the general case.