KMB, 25/10/01

M1F Foundations of analysis—Problem Sheet 4.

1)

(a) Find the real and imaginary parts of $(1+i)^{100}$.

(b) If you know the binomial theorem, then use part (a) to evaluate the sum $\binom{100}{0} - \binom{100}{2} + \binom{100}{4} - \binom{100}{6} + \ldots + \binom{100}{100}$.

(c) If you want a challenge, find a purely combinatorial proof of (b) (that is, don't use (a) or indeed anything about the complex numbers; just use facts about binomial coefficients.)

2) By considering $\frac{1+i}{\sqrt{3}+i}$, or otherwise, show that $\cos(\pi/12) = \frac{\sqrt{6}+\sqrt{2}}{4}$. Is this irrational?

3)

(a) Find all ten 10th roots of i. Which one is nearest to i in the Argand diagram?

(b) Let z be a non-zero complex number. Prove that the three cube roots of z are the vertices of an equilateral triangle on the Argand diagram.

4) Here's another cosine evaluated explicitly. Let $\zeta = e^{2\pi i/5}$ be one of the 5th roots of 1.

a) Show that $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$.

b) Let $\alpha = \zeta + \zeta^4$, and let $\beta = \zeta^2 + \zeta^3$. Prove that α and β are the roots of the polynomial $X^2 + X - 1$.

c) Deduce that $\cos(2\pi/5) = \frac{\sqrt{5}-1}{4}$.

5) Here's one way of getting the formula for the roots of a general cubic polynomial $ax^3 + bx^2 + cx + d$, a non-zero.

a) Firstly check this lemma. If r and s are two complex numbers, and we know their sum and their product, then we can work out r and s. Hint: either write down a cunning quadratic equation, or work out the square of $(r - s)$ and proceed from there.

b) Back to the general cubic. By "completing the cube", and dividing out by a, show that it suffices to find a formula for the roots of the cubic $x^3 + Ax + B$.

c) Let's seek solutions of the form $x = M + N$. Check that now the equation we're trying to solve becomes $M^3 + N^3 + 3MNx + Ax + B = 0$. Deduce that if we can find complex numbers M and N such that $M^3 + N^3 = -B$ and $3MN = -A$ then we can construct a root of the cubic.

d) Note that $M^3N^3 = -A^3/27$. Hence we know what we want the sum of M^3 and N^3 to be, and we also know what we want the product to be. By part (a), evaluate M^3 and N^3 (don't worry that we don't know which if M^3 and which is N^3, it won't matter). Now deduce a formula for x. How many solutions does this method yield?