KMB, 19/10/01.

M1F Foundations of Analysis—Problem Sheet 3, hints and solutions.

1)

(a) By (2) applied to $x > 0$ and $t = y$ we deduce $x + y > y$. We know that $y > 0$, and so by (4) we deduce $x + y > 0$. One mark.

(b) We are given $y > x$ and $x > 0$ and so by (4) we deduce $y > 0$. By fact (G) from the course, $1/y > 0$. All that is left is to prove that $1/x > 1/y$. One way to do this would be to observe that by (3) we have $xy > 0$ and so by (G) we have $1/(xy) > 0$. Now by (D) applied to the inequality $y > x$ and the constant $c = 1/(xy)$ we deduce that $1/x > 1/y$. Two marks.

(c) By (5) we have an integer $n > x$ and we are given $x > 0$ hence by (b) we have $1/x > 1/n > 0$. But $1/n$ is rational so setting $q = 1/n$ will work. Two marks.

2) This is pretty tricky, I think. It helps to know

Lemma. (strengthening of axiom 5) If x is any real then there is an integer n with $n - 1 \leq x < n$.

Proof. By (5) there exists an integer n_1 with $x < n_1$. Similarly there exists n_2 with $-x < n_2$ and hence $-n_2 < x$ by (2). So $-n_2 < x < n_1$. There are only finitely many integers m with $-n_2 \leq m \leq n_1$, so let this finite set of integers be S. We know that S contains some integers m with $m > x$, for example $m = n_1$. Let n be the smallest of these integers. Then $n > x$, so $n \neq -n_2$, so $n - 1 \in S$, and by definition of n we have $x \geq n - 1$, which is what we wanted. □

Now onto the question. We have $b > a$ so by (2) we have $b - a > 0$. By Q1 (c) we deduce that there is a rational q with $b - a > q > 0$. By (G) we have $1/q > 0$ and so by (D) we have $b/q - a/q = (b-a)/q > 1$. So by (2) we see that $b/q > 1 + a/q$. By the lemma above there is an integer n with $n > a/q \geq n - 1$ and hence by (2) we have $a/q + 1 \geq n$. By (4) we deduce that $b/q > n$ so $b/q > n > a/q$ and hence $b > nq > a$ and $c = nq$ will do.

3*)

(a) if $x = 3^{1/3}$ and $y = 2^{1/2}$ then $x^6 = 3^2 = 9$ and $y^6 = 2^3 = 8$ and hence $y^6 < x^6$, so $y < x$ as both are positive. Two marks, but if you used a calculator to work out both sides to a few decimal places, then lose a mark unless you proved that the decimal expansions really were correct to the order of magnitude you need: for example it would be OK to say that $1.43^2 = 2.04\ldots > 2$ but $1.43^3 = 2.924\ldots < 3$ so $2^{1/2} < 1.43 < 3^{1/3}$, but it wouldn't be OK just to state without proof that $2^{1/2} = 1.41\ldots$ "because my calculator says it is".

(b) $100^{10000} = (10^2)^{10000} = 10^{20000}$ and 10000^{100} is a measly $(10^4)^{100} = 10^{400}$, so the first is bigger. Two marks.

(c) 2^{11} and $2^{2^{21}}$ respectively. Two marks.

4*)

(a) Using the method I explained in lectures, the question is equivalent to asking for x such that $2x + 1/x - 3 < 0$, that is, $(2x^2 - 3x + 1)/x < 0$. The LHS factors as $(2x - 1)(x - 1)(1/x)$ so the "critical points" where sign-changes occur are $x = 0$, $x = 1$ and $x = 1/2$; within the four ranges of reals defined by these three points, we count how many terms are positive, and after some work we get that the regions which work are $x < 0$ or $1/2 < x < 1$. Surely you can do these by now! Two marks.

Lots of ways to lose marks here—if you don't point out that the question is *equivalent* to $(2x^2 - 3x + 1)/x < 0$ then your argument is not watertight because you could have introduced spurious solutions, and so you should lose a mark (unless you checked your answer at the end). If you are sloppy with ends of inequalities then you should lose another one per slip—there's no room for sloppiness in pure maths!

(b) As I explained in lectures, you get two distinct real roots precisely when $t^2/4 - 3 > 0$ because the equation is equivalent to $(x + t/2)^2 = t^2/4 - 3$. The inequality holds exactly when $t^2 > 12$, that is, when $(t + \sqrt{12})(t - \sqrt{12}) > 0$, and this will hold precisely when $t > \sqrt{12}$ or $t < -\sqrt{12}$. Three marks.

5*)

(a) If $z = a + ib$ and $w = c + id$ then one checks easily that the LHS and the RHS are both $(a + c) - i(b + d)$. One measly mark.

(b) If $z = a + ib$ then by definition $|z| = \sqrt{a^2 + b^2}$ so $|z|^2 = a^2 + b^2$ and multiplying out $z.\overline{z}$ also gives $a^2 + b^2$. One more measly mark.

(c) We know that $z = \cos\theta + i\sin\theta$ and so $\overline{z} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = (r, -\theta)$.

6) Hint: what is the largest number you can get by adding together the 1000th powers of 1-digit numbers? And what is the smallest 1000-digit number?