KMB, 18/10/01.

M1F Foundations of analysis—Problem Sheet 3.

This is assessed coursework. Please hand in solutions to the starred questions (i.e., to questions 1, 3, 4, 5) at the end of the lecture on Monday 29th October.

1*) Recall the following 5 axioms about the ordering $<$ on the real numbers, that were mentioned in lectures (here x, y, z, t are all reals):

(1) For all x, exactly one of $x > 0$ or $0 > x$ or $x = 0$ is true.
(2) If $x > y$ then $x + t > y + t$.
(3) If $x > 0$ and $y > 0$ then $xy > 0$.
(4) If $x > y$ and $y > z$ then $x > z$.
(5) For all x there is an integer n with $n > x$.

Using *only* these axioms about $<$, and facts deducible from them (for example, you may use (A)–(H) from lectures), prove (a)–(c) below. State, at each step of your proofs, which axiom or fact you are appealing to.

(a) If $x > 0$ and $y > 0$ then $x + y > 0$.
(b) If $y > x > 0$ then $1/x > 1/y > 0$.
(c) If $x > 0$ then there is a rational number q with $0 < q < x$.

2) Prove that if $a < b$ are reals, then there is a rational number c with $a < c < b$, just using the axioms.

3*) (In this and subsequent questions, you may assume anything reasonable about $<$ and do not have to reduce everything to the axioms stated in Q1.)

(a) Which is bigger, the cube root of 3 or the square root of 2? (Hint: a calculator-free proof exists, involving getting rid of all the fractional powers).
(b) Which is bigger, 100^{10000} or 10000^{100}?
(c) What's the square root of 2^{22}? What's the square root of $2^{2^{22}}$?

4*)
a) Find the set of reals $x \neq 0$ such that $2x + \frac{1}{x} < 3$.
b) Find the set of reals t such that the equation $x^2 + tx + 3 = 0$ has two distinct real solutions $x = x_1$ and $x = x_2$.

5*)
(a) If z and w are complex numbers then prove that $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}.\overline{w}$.
(b) If z is complex then prove that $|z|^2 = z.\overline{z}$.
(c) If $z = (r, \theta)$ in polar form, then prove that $\overline{z} = (r, -\theta)$.

6) The number 371 is a 3-digit number which is equal to the sum of the cubes of its digits! Prove that there is no 1000-digit number equal to the sum of the 1000th powers of its digits (using a calculator is OK for this one—the trick is working out what to calculate!)