KMB, 09/10/01

M1F Foundations of Analysis—Problem Sheet 2 hints and solutions

1)

(a) $\sqrt{3}$ is irrational, unsurprisingly, and the same proof works as for $\sqrt{2}$, changing appropriate "2"s to "3"s. Points to note are: things like "even" and "odd" should be replaced by "is a multiple of 3" and "is not a multiple of 3". The key point is that if 3 divides n^2 then 3 divides n, but I proved this in lectures.

(b) is false, and here is a counterexample: $0.\sqrt{2} = 0$.

2)

(a) $\sqrt{2}+\sqrt{3/2}$ is irrational—proof by contradiction. If it were rational, then its square would be too, and hence (putting the rationals all on one side) we deduce that $2\sqrt{3}$ is rational, hence (multiplying by $1/2$) that $\sqrt{3}$ is rational, contradicting Q1(a).

(b) If $1 + \sqrt{2} + \sqrt{3/2}$ were rational then subtracting one we deduce that $\sqrt{2} + \sqrt{3/2}$ is rational, contradicting part (a).

(c) Standard manipulation shows that $2\sqrt{18} - 3\sqrt{8} + \sqrt{4} = 2$ which is certainly rational.

(d) By contradiction. Let's assume for a contradiction that $\sqrt{2}+\sqrt{3}+\sqrt{5} = a/b$ is rational. Rewrite the equation as $\sqrt{2} + \sqrt{3} = a/b - \sqrt{5}$. Now square both sides and deduce that $2\sqrt{6} = c/d - 2(a/b)\sqrt{5}$ with c/d some rational number. We're down to two square roots: put them on the same side and see that $2\sqrt{6} + 2(a/b)\sqrt{5}$ is rational. Square up and sort out the rationals again to deduce that $8(a/b)\sqrt{30}$ is rational. But $\sqrt{30}$ is clearly not an integer and hence it is irrational. So the only way that $8(a/b)\sqrt{30}$ can be rational is if $8a/b = 0$. But this implies that $a/b = 0$ and hence that $\sqrt{2} + \sqrt{3} + \sqrt{5} = 0$, which is not true because the left hand side is positive. Hence we have our contradiction and we are done.

3)

(a) If $x = 1.234234234\ldots$ then $1000x = 1234.234234234\ldots$ and hence $999x = 1233$. Hence $x = 1233/999$ (which equals $137/111$ as it happens).

(b) a is irrational. One can prove this by contradiction. If a were rational then its decimal expansion would ultimately be periodic. Say the periodic part of a had length N, and that

$$a = 0.a_1a_2a_3\ldots a_{n-1}a_n\overline{b_1b_2\ldots b_N}.$$

We can evaluate b_1 to b_N by looking at the behaviour of a's decimal expansion, as long as we look sufficiently far. So consider the decimal expansion of a somewhere after the nth digit (so the expansion is recurring), where the blocks of zeros in the decimal expansion of a are much greater in size than N. We note that somewhere within one of these huge blocks of zeros must be an occurrence of the period $b_1b_2\ldots b_N$. Hence all of the b_i must be 0. But this means that all the entries in the decimal expansion of a are zero after some point, which is not true because infinitely many are 1. Contradiction.

(c) $x = 0.12301230\ldots$ and the standard method gives $9999x = 1230$, so $x = 1230/9999 = 410/3333$ is rational.

4) 7 is the smallest counterexample (Optional exercise: prove that if $n > 0$ is of the form $8m + 7$ then n is a counterexample. But also 28 is a counterexample. Do some experiments on a computer to guess the general result as to precisely which integers are the sum of three squares).

5) If $x = \frac{100}{9899}$ then it's easy to check that $100 + x + 100x = 10000x$. Writing this out in full, using the fact that $x = 0.0101\ldots$, and spacing it out a bit to make the point, gives

$$\begin{array}{cccccccccc} & 100 & & & & & & & \ldots \\ + & 0 & . & 01 & 01 & 02 & 03 & 05 & \ldots \\ + & 1 & . & 01 & 02 & 03 & 05 & 08 & \ldots \\ = & & & & & & & & \\ & 101 & . & 02 & 03 & 05 & 08 & 13 & \end{array}$$

If you think about what's going on here, you'll see that, at least near the beginning of the decimal expansion of x, the numbers formed by looking at pairs of digits are being forced to obey the rule defining the Fibonacci sequence. Of course eventually one has to start carrying digits and the whole thing goes wrong, but it does work for a while, and if you were to try, say, $x = 1000/998999$ then it would work for even longer.