KMB, 09/10/01

M1F Foundations of Analysis—Problem Sheet 2

Questions marked with a † might be harder than the rest.

1) Decide whether the following statements are true or false, and write down a proof or a counterexample as appropriate.

(a) $\sqrt{3}$ is irrational.
(b) The product of a rational and an irrational number is always irrational.

2) Are the following numbers rational or irrational? Proofs required.

(a) $\sqrt{2}+\sqrt{3/2}$ (hint: if it were rational then its square would be too).
(b) $1+\sqrt{2}+\sqrt{3/2}$.
(c) $2\sqrt{18}-3\sqrt{8}+\sqrt{4}$.
(d)† $\sqrt{2}+\sqrt{3}+\sqrt{5}$ (hint: use the fact that if $n \geq 0$ is an integer, and $\sqrt{n}$ is not an integer, then $\sqrt{n}$ is irrational.)

3)

(a) Write the recurring decimal $1.\overline{234}$ as a rational.
(b) Let $a = 10^{-1}+10^{-2}+10^{-4}+10^{-8}+10^{-16}+\ldots$ (the exponent doubles each time). Is a rational or irrational? Proof required!
(c) Let the decimal $x = 0.a_1a_2a_3a_4\ldots$ be defined by letting a_n be the remainder when n is divided by 4. Is x rational or irrational?

4) Proof or counterexample: every positive integer is the sum of three squares.

5) The Fibonacci sequence starts with the terms $1, 1$ and then proceeds by letting the next term be the sum of the two previous terms. So the series starts $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. With this in mind, consider the decimal expansion of $x = \frac{100}{9899}$; it is $0.0101020305081321 34\ldots$. Note how the Fibonacci sequence is living inside this decimal expansion! Can you explain this phenomenon? Do you think it continues forever?

[Hint: check that $100 + x + 100x = 10000x$, and write down the decimal expansion of this equation.]