KMB, 28/09/01

M1F Foundations of analysis— Problem Sheet 1, hints and solutions.

1*) TFTFTFFTFF. 1/2 mark each.

2)
(a) $\{n \in \mathbb{Z} : n^2 > 1000\}$
(b) $\{x \in \mathbb{R} : x^2 + x + 1 < 0\}$. This set happens to be empty—complete the square to see why.

3*) 1 is an element of A and also of B, but not of C. So by definition of union and intersection, we see that (a) and (c) are true, but (b) and (d) are false. 1 mark each.

4*)
(a) By contradiction. Let's assume $\sqrt{2} + \sqrt{6} \geq \sqrt{15}$. Then, because both sides are positive, we may square and deduce that $8 + 2\sqrt{12} \geq 15$ and hence $2\sqrt{12} \geq 7$. Squaring again we deduce that $48 \geq 49$, a contradiction. So the original statement is proved. Three marks for this—but no marks for assuming the statement you have to prove and then deducing something which is obviously true—if you do this then you have proved nothing. No marks either for writing down a list of mathematical statements without any indication of which of them imply the others!
(b) Again, by contradiction. If $q = a/b$ is a non-zero rational, with a and b integers, then a must be non-zero. If r is an irrational then the question asks us to prove that qr is irrational. But if qr were rational then one could write $qr = c/d$ with c and d integers, and then one sees $r = bc/ad$ (note that we can divide by a because it's non-zero). Three marks for this, but lose a mark if you divided by a without noting that it was non-zero.

5) One can disprove these statements by giving counterexamples.
(a) We showed in lectures that $\sqrt{2}$ was irrational, but $\sqrt{2}.\sqrt{2} = 2$ is visibly rational.
(b) It is a consequence of a result in lectures that $\sqrt{2}$ and $1+\sqrt{2}$ and $2+\sqrt{2}$ are all irrational—but $\sqrt{2}.(1 + \sqrt{2}) = 2 + \sqrt{2}$.

6*) We prove this by contradiction. Suppose, for a contradiction, that $x > 0$. Set $y = x/2$. Then $y > 0$ and hence by assumption we have $x \leq y$. But this implies that $2x \leq x$ and hence $x \leq 0$, a contradiction. Hence our original assumption was false and $x \leq 0$. Five marks. Note that this question should firmly dispel from your mind the totally incorrect thought that there is a real number whose decimal expansion is $0.000\ldots 1$, with infinitely many zeroes before the 1. There is no "smallest positive real number" because given any positive real number, one can divide it by two and get a smaller one.

7) If it helps, assume (or prove) that if $m \geq 0$ is an integer, and $\sqrt{m}$ is rational, then $\sqrt{m}$ is also an integer. IF you need any more hints, ask your tutor.