Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2003

This paper is also taken for the relevant examination for the Associateship.

M1F FOUNDATIONS OF ANALYSIS

DATE: Tuesday, 20th May 2003 TIME: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Define what it means for a real number to be

   (i) rational

   (ii) irrational

   (b) Prove each of the following statements.

   (i) If a is rational and b is irrational, then $a+b$ is irrational.

   (ii) If $a \neq 0$ is rational and b is irrational, then ab is irrational.

   (iii) Between any two distinct rational numbers there is an irrational number.

   (c) Explain *briefly* why there do not exist positive integers p and q such that $10^{p/q} = 2$.

   (d) Let x be the real number whose decimal expansion is

$$x = 0 \cdot a_1 a_2 a_3 \dots$$

   where $a_n = 1$ if n is a perfect square and $a_n = 0$ otherwise.

   Say whether x is rational, justifying your answer briefly.

   [Hint: There are $2m$ integers between m^2 and $(m+1)^2$.]

2. (a) Write the complex number $(1+i)$ in polar form.

   (b) Calculate $(1+i)^{200}$.

   (c) State the binomial theorem (which works for both real and complex numbers).

   (d) Deduce that

$$2^{100} = \sum_{k=0}^{100} \binom{200}{2k} (-1)^k.$$

   (e) Write down the fifth roots of unity, i.e. those z such that $z^5 = 1$, and draw them in the complex plane.

   (f) Prove that if $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$, then ω is not a real number.

3. (a) State the principle of strong induction.

   (b) Define what it means for an integer $p \geq 2$ to be *prime*.

   (c) Prove that every positive integer $m \geq 2$ is a product of prime numbers.

   (d) Prove that there are infinitely many prime numbers.

   (e) Suppose $m = p_1 \dots p_r$, where the p_i are prime and $p_i \leq p_{i+1}$ for $i = 1, \dots, r-1$. Prove that $p_r \geq m^{1/r}$.

4. (a) Define what it means to say that $U \in \mathbb{R}$ is the *least upper bound* (LUB) for a non-empty set $S \subseteq \mathbb{R}$. (In lectures we wrote $U = \mathrm{LUB}(S)$.)

(b) Let S be the set of real numbers between 0 and 1 whose decimal expansion contains no nines. What is $\mathrm{LUB}(S)$? [No proof is required.]

(c) For each of the following statements, either prove that it is correct or give a counterexample.

(i) Every non-empty subset of $\mathbb{R}$ has a least upper bound.

(ii) If S contains no rational numbers, then $\mathrm{LUB}(S)$ is irrational.

(iii) Every real number is least upper bound for some set of rational numbers.

(iv) Let $S, T \subset \mathbb{R}$ be non-empty subsets and assume that $c = \mathrm{LUB}(S)$ and $d = \mathrm{LUB}(T)$ exist. Define $P = \{s - t \mid s \in S,\ t \in T\}$. Then $\mathrm{LUB}(P) = c - d$.

(v) Let $S = \{x \in \mathbb{R} \mid x^2 \leq 2\}$. If $\mathrm{LUB}(T) = \mathrm{LUB}(S)$, then T is either finite or uncountable.

5. (a) Prove that there do not exist positive integers n and m such that $n^2 \equiv m^2 + 2 \mod 4$.

(b) Find a pair of integers λ and μ such that $21\mu + 13\lambda = 1$ with $100 < \lambda < 150$.

(c) Use the fact that $1001 = 13 \times 77$ to work out the remainder when 8007004003005008 is divided by 13.

(d) We define an equivalence relation on $\mathbb{Z}$ by $n \sim m$ if $m \equiv n \mod 7$. Let S be the set of equivalence classes of this relation. How many distinct maps are there from the set $\{1, 2, 3\}$ to S? How many of these maps are injective (i.e. 1-to-1)? How many are surjective (i.e. onto)?