UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSC/MSCI EXAMINATION (MATHEMATICS) MAY-JUNE 2001

This paper is also taken for the relevant examination for the Associateship.

M1F FOUNDATIONS OF ANALYSIS

DATE: *Tuesday 22 May 2001* TIME: *10am - 12 noon*

Credit will be given for all questions attempted, but extra credit will be given for complete, or nearly complete, answers.

Calculators may not be used.

1. (a) Prove that if n is an integer, then n is even if and only if n^3 is even.

(b) Prove that $\sqrt[3]{2}$ and $\sqrt[3]{4}$ are irrational.

(c) Prove that $\sqrt[3]{2} + \sqrt[3]{4}$ is irrational.

Hint for Part (c): If $t = \sqrt[3]{2} + \sqrt[3]{4}$ then consider $t^2 - t$.

2. Sketch the proof that if a convex polyhedron has F faces, E edges and V vertices, then $F - E + V = 2$.

3. *In this question, standard facts about inequalities may be assumed.*

(a) If S is a set of reals, what does it mean to say that a real number x is a *least upper bound* for S?

(b) If $S = \{s_1, s_2, \ldots, s_n\}$ has only finitely many elements, with $s_1 < s_2 < \ldots < s_n$, then prove that s_n is the least upper bound for S.

(c) If X is a set of reals, we write $LUB(X)$ for its least upper bound. Does there exist a set S such that

$$LUB(\{x^2 : x \in S\}) = 4$$

and $LUB(\{x^3 : x \in S\}) = 1$? Justify your answer.

(d) Does there exist a set T such that

$$LUB(\{x^2 : x \in T\}) = 4$$

and $LUB(\{x^3 : x \in T\}) = 27$? Justify your answer.

4. (a) What does it mean for a binary relation on a set X to be

(i) reflexive;

(ii) symmetric;

(iii) transitive;

(iv) an equivalence relation?

(b) If $\sim$ is an equivalence relation on a set X, define an *equivalence class* and show that the equivalence classes give a partition of X.

(c) How many binary relations are there on the set $\{1,2\}$? And how many are equivalence relations?

5. (a) What does it mean for a set to be *countable*?

(b) Prove that $\mathbb{Q}$ is countable.

(c) Prove that the set of all subsets of $\mathbb{Q}$ is not countable.