I M P E R I A L C O L L E G E O F S C I E N C E
T E C H N O L O G Y & M E D I C I N E

DEPARTMENT OF MATHEMATICS

First Year Test Paper - January 2001

M1F FOUNDATIONS OF ANALYSIS

Date: Monday 8 January 2001

Time: 10.15am - 11.45am

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers. The question in Section A will be worth 1½ times as many marks as either question in Section B. Answer each question in a separate book. Write your name and the question number prominently on the front of each book.

SECTION A

In each part of this question, just write down your answers. No justification is required.

1. PART I

Say whether the following statements are true or false:

(i) If x is rational and y is irrational, then $x + y$ must be irrational.

(ii) The set $\left\{ x \in \mathbb{R}: \frac{x+1}{x^2+1} \leq 3 \right\}$ is bounded below.

(iii) The decimal expansion of $\frac{1}{7}$ is recurring, with a period of length 6.

(iv) $f: \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = n^2$, is an injection.

(v) $10^{2001} - 3^{2001}$ is a multiple of 7.

(vi) If A and B are infinite sets, then $A \cap B$ is also infinite.

PART II

(i) The real part of $(\sqrt{3} + i)^{2000}$ is

(a) -2^{1999}; (b) -2^{1000}; (c) 2^{1000};

(d) 2^{1999}; (e) something else.

(ii) Which is bigger, $2^{1/2}$ or $3^{1/3}$?

(a) $2^{1/2}$; (b) $3^{1/3}$; (c) they are equal.

QUESTION 1 IS CONTINUED OVER...

(iii) The binary relation $\sim$ on $\mathbb{R}$ is defined by $a \sim b$ if $a + b < 0$. Which of the following describes the properties of this relation?

(a) $\sim$ is not symmetric, transitive or reflexive;

(b) $\sim$ is symmetric but not transitive or reflexive;

(c) $\sim$ is symmetric and transitive, but not reflexive;

(d) $\sim$ is symmetric, transitive and reflexive.

(iv) My child has 20 toys. Of these, 6 need new batteries and 8 have fallen behind the sofa. If 10 toys neither need new batteries nor are behind the sofa, how many toys behind the sofa need new batteries?

(a) 2; (b) 4; (c) 6; (d) 8; (e) 10.

(v) One of the toys mentioned above is a convex polyhedron with 90 edges and 60 vertices. How many faces does it have?

(a) 28; (b) 30; (c) 32; (d) some other number.

(vi) The number of surjective functions from a set of size 3 to itself is

(a) 0; (b) 1; (c) 2; (d) 6;

(e) some other number.

(vii) The coefficient of $x^2y^2z^2$ in $(x+y+z)^6$ is

(a) 20; (b) 30; (c) 90; (d) 100;

(e) some other number.

SECTION B

2. (a) Prove that if n is an integer, and n^2 is a multiple of 3, then n is a multiple of 3.

(b) Prove that $\sqrt{3}$ is irrational.

(c) Prove that $\cos(5\pi/12)$ is irrational. Hint: Consider the complex number $(1+i)(\sqrt{3}+i)$.

3. Let S be a set of real numbers. Define what it means for S to be *bounded above*. Define what it means for a real number u to be

(i) an upper bound for S;

(ii) a least upper bound for S.

If $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ then prove that S has a least upper bound, and find this bound.

Prove that if S is a set of reals with a least upper bound u, then the set $T = \{s+1: s \in S\}$ has a least upper bound, and find this bound.

If S is a set of reals which is bounded above, is the set $U = \{s^2: s \in S\}$ also bounded above? Justify your answer.