IMPERIAL COLLEGE OF SCIENCE
TECHNOLOGY & MEDICINE

DEPARTMENT OF MATHEMATICS

First Year Test Paper - January 1999

M1F FOUNDATIONS OF ANALYSIS

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Date: Tuesday 12 January 1999

Time: 10.15am - 11.45am

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers. The question in Section A will be worth 1½ times as many marks as either question in Section B. Answer each question in a separate book. Write your name and the question number prominently on the front of each book.

SECTION A

In each part of this question, just write down your answer. No justification is required.

1. PART I

Say whether the following statements are true or false:

(i) For a real number y, $y^3 - y < 0$ only if $y < 1$.

(ii) If $S = \{3, \{7, \{2\}\}, \{8\}\}$, then $\{2\} \in S$.

(iii) The least upper bound of a set of rational numbers is always rational.

(iv) $(\sqrt{3} + i)^6 = 64$.

(v) The function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ defined by

$$f(m,n) = 2^m 3^{-n} \qquad (m,n \in \mathbb{Z})$$

is 1-1.

(vi) The relation $\sim$ defined on $\mathbb{R}$ by

$$a \sim b \Leftrightarrow a = b \text{ or } -b \qquad (a,b \in \mathbb{R})$$

is an equivalence relation.

QUESTION 1 IS CONTINUED OVER...

PART II

(i) The greatest lower bound of the set

$$\{x \in \mathbb{R}: x^4 + x^3 + x^2 + x < 0\}$$

is

(a) 0 (b) -1 (c) i (d) does not exist.

(ii) You are given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. The roots of the cubic equation

$$8x^3 - 6x + 1 = 0$$

are

(a) $\cos\frac{\pi}{9}, \quad \cos\frac{5\pi}{9}, \quad \cos\frac{7\pi}{9}$

(b) $\cos\frac{2\pi}{9}, \quad \cos\frac{4\pi}{9}, \quad \cos\frac{8\pi}{9}$

(c) $\cos\frac{\pi}{3}, \quad \cos\frac{5\pi}{3}, \quad \cos\frac{7\pi}{3}$

(d) $\cos\frac{2\pi}{3}, \quad \cos\frac{4\pi}{3}, \quad \cos\frac{8\pi}{3}$.

(iii) If $d = \text{hcf}(1209, 221)$, then $d = 1209s + 221t$ for precisely one of the following pairs s, t. Which?

(a) $s = 2, \quad t = -11$ (b) $s = 1, \quad t = -5$

(c) $s = -2, \quad t = 11$ (d) $s = 4, \quad t = -25$.

QUESTION 1 IS CONTINUED OVER...

(iv) One of the following congruence equations does not have a solution $x \in \mathbb{Z}$. Which?

(a) $x^2 + 1 \equiv 0 \bmod 13$

(b) $999x \equiv 9 \bmod 9999$

(c) $999x \equiv 9 \bmod 999999$.

(v) If $2^{90} \equiv r \bmod 7$ with $0 \leq r \leq 6$, then r is

(a) 1 (b) 2 (c) 4 (d) 6.

(vi) The coefficient of y^2 in $\left[y^2 - \dfrac{1}{y^3}\right]^6$ is

(a) 6 (b) -6 (c) -20 (d) 15 (e) 1.

(vii) Let $S = \{1,2,3\}$ and $T = \{a,b\}$. The number of functions from $S \rightarrow T$ which are onto, is

(a) 9 (b) 0 (c) 6 (d) 3 (e) 4.

SECTION B

2. (i) Prove that for all integers $n \geq 1$,

$$1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2} .$$

(ii) Find all complex solutions of the equation

$$z^5 - 1 - i = 0 .$$

Which of these solutions is closest to the imaginary axis?

3. Define what is meant by a countable set.

(a) Prove that $\mathbb{Z}$ is countable.

(b) Prove that $\mathbb{Q}^+$ (the set of positive rationals) is countable.

(c) Let S be the set consisting of all infinite sequences $a_1 a_2 a_3 \ldots$, where each a_i is 0 or 1 (so a typical member of S is 0100110111001...). Prove that S is uncountable.