Last time.

We saw some local examples. We saw how integrating the product of the characteristic function of $\mathbf{Z}_p$ and the character $|\cdot|^s$ over the group $\mathbf{Q}_p^\times$ gave us (for $Re(s) > 0$) the function $1/(1-p^{-s})$. And we saw that integrating $e^{-\pi x^2}$ times $|x|^s$ on $\mathbf{R}^\times$ gave us $\pi^{-s/2}\Gamma(s/2)$.

We saw that some relatively simple local calculation showed that for f and g well-behaved functions on K, $\zeta(f,c)\zeta(\hat{g},\hat{c}) = \zeta(g,c)\zeta(\hat{f},\hat{c})$ for $0 < Re(c) < 1$, and hence that $\zeta(f,c)$ had a meromorphic continuation to all of $\mathbf{C}$.

We finished by seeing the construction of the restricted product of topological groups – if G_i are locally compact topological groups and, for all but finitely many i, H_i is a compact open subgroup of G_i, then the restricted product $\prod_i' G_i$ of the G_i is also locally compact. Recall that this product is elements (g_i) such that $g_i \in H_i$ for all but finitely many i.

The adeles and ideles.

Let k be a number field. If P is a (non-zero) prime ideal of the integers of k then completing k with respect to the P-adic norm gives us a field k_P with open compact ring of integers R_P. For example, completing $\mathbf{Q}$ with respect to the p-adic norm gives us $\mathbf{Q}_p$ with open compact subring $\mathbf{Z}_p$. Completing with respect to an infinite place $\tau : k \to \mathbf{C}$ gives us an archimedean completion, isomorphic to either the reals or the complexes. There are only finitely many of these!

Hence we are in a situation where we can apply the restricted product situation. We can consider the restricted product of all the k_v with respect to their subrings R_v, giving us the ring $\mathbf{A}_k$ of *adeles of* k, or we can consider the restricted product of all the $k_v^\times$ with respect to the open compact unit groups $R_v^\times$, giving us the ideles $\mathbf{A}_k^\times$, the units in $\mathbf{A}_k$.

Example: $\mathbf{A}_\mathbf{Q}$ is the subring of $\mathbf{Q}_2 \times \mathbf{Q}_3 \times \mathbf{Q}_5 \times \mathbf{Q}_7 \times \cdots \times \mathbf{R}$ consisting of elements (x_v) such that $x_p \in \mathbf{Z}_p$ for all but finitely many p.

More definitions: $\mathbf{A}_k$ naturally splits up as a product $\mathbf{A}_k^f \times \mathbf{A}_{k,\infty}$, where $\mathbf{A}_k^f$, the *finite adeles*, are the restricted product over the non-archimedean factors, and $\mathbf{A}_{k,\infty}$, the infinite adeles, are the product over the finitely many archimedean completions of k.

Relationship with k.

Note that there is a natural map from k to $\mathbf{A}_k$! Because an element of k only has finitely many primes occurring in its denominator, and so is integral at all but finitely many of the finite places. Not only that, but actually the induced (subspace) topology on k is actually the discrete topology. Indeed $\prod_P R_P$ is an open subgroup of the finite adeles of k, and the intersection of k with this open subgroup in $\mathbf{A}_k^f$ is equal to the element of k which have no primes at all in their denominator, and hence equals R, the integers of k. It is well-known that R is a lattice in $\mathbf{A}_{k,\infty} = k \otimes_\mathbf{Q} \mathbf{R}$, so we can choose a sufficiently small open ball around 0 in the infinite adeles that contains no other integer, and the product of this open ball and $\prod_P R_P$ is an open set in $\mathbf{A}_k$ whose intersection with k is just $\{0\}$. Becuase $\mathbf{A}_k$ is a topological group under addition, this is enough.

Moreover, we can also see that the quotient $\mathbf{A}_k/k$ is compact. Indeed, given an adele a, a trick involving the Chinese Remainder Theorem shows that there is some $\lambda \in k$ such that $a - \lambda$ is integral at every place, giving us a natural surjection from $\mathbf{A}_k/k$ to the compact space $\mathbf{A}_{k,\infty}/R$ whose kernel is $\prod_P R_P$ and compactness follows.

Similarly there is a map $k^\times \to \mathbf{A}_k^\times$ and again the image of $k^\times$ is discrete, although this time you need the proof of Dirichlet's Unit Theorem to show that the units are a lattice in $(k \otimes_{\mathbf{Q}} \mathbf{R})^\times$. The quotient is not compact though, because of some issue involving norms, which I will come to now.

Global norm.

Now recall this slightly disconcerting way of defining functions on restricted products. If we have a restricted product $\prod_i' G_i$ of abelian groups, and group homomorphisms $c_i \to \Gamma$ for some abelian group Γ, such that H_i is in the kernel of c_i for all but finitely many i, then (even if Γ has no topology!) we can consider $c := \prod_i c_i$, something which looks like an infinite product, but which makes sense as a function on $\prod_i' G_i$ because for any element of the product, all but finitely many of the c_i will evaluate to 1 so it's actually a finite product!

For example, if K is a finite extension of $\mathbf{Q}_p$ then the local norm $|\cdot| : K^\times \to \mathbf{R}_{>0}$ is trivial on the units of K, and hence for k a number field we can consider the product of these local norms on $\mathbf{A}_k^\times$, giving us the global norm

$$|\cdot| : \mathbf{A}_k^\times \to \mathbf{R}_{>0}$$

sending (x_v) to the finite product $\prod_v |x_v|$. Unsurprisingly, the norm of an idele is the factor by which multiplication by the idele scales the additive Haar measure on the adeles (because this was true locally). Unsurprisingly also, the restriction of $|\cdot|$ to $k^\times$ is trivial (because it is a canonical norm so it can't be any of the interesting ones!).

The multiplicative analogue of $\mathbf{A}_k/k$ being compact is the fact that the image of $k^\times$ in $\mathbf{A}_k^\times$ is in the kernel J of the global norm map, and the quotient $J/k^\times$ is compact. In contrast to the additive result, which just uses basic facts about rings of integers, this really uses something: it is equivalent to the finiteness of the class group and the fact that the unit group has the rank it is supposed to have.

Self-duality.

Here's something else we saw in the local setting which translates to the global setting. If K is a finite extension of $\mathbf{Q}_p$ or $\mathbf{R}$ then we defined a map $\Lambda : K \to \mathbf{R}/\mathbf{Z}$, a non-trivial additive group homomorphism, and noted that the resulting pairing $K \times K \to S^1$ sending (x, y) to $e^{2\pi i \Lambda(xy)}$ identified K with $\widehat{K}$. We even carefully chose a Haar measure on K such that $\hat{\hat{f}}(x) = cf(-x)$ with $c = 1$. Unsurprisingly, if K is non-archimedean and x and y are both in R_K, the integers of K, then $\Lambda(xy) = 0$. This means that the infinite-looking sum $\Lambda((x_v), (y_v)) := \sum_v \Lambda_v(x_v y_v)$ is actually finite, and a well-defined function on $\mathbf{A}_k$, giving us a non-trivial map $\mathbf{A}_k \to \widehat{\mathbf{A}_k}$ which can be checked to be an algebraic and topological isomorphism.

The zeta function is back!

Let's define a function on the ideles $\mathbf{A}_{\mathbf{Q}}^\times$ thus: for p a prime number, define f_p on $\mathbf{Q}_p$ to be the characteristic function of $\mathbf{Z}_p$. Define f_∞ on $\mathbf{R}$ to be $e^{-\pi x^2}$. Define $f : \mathbf{A}_{\mathbf{Q}}^\times \to \mathbf{C}$ by $f((g_v)) = \prod_v (f_v(g_v))$ (a finite sum). Now consider the function

$$s \mapsto \int_{\mathbf{A}_{\mathbf{Q}}^\times} f(x)|x|^s \mathrm{d}\mu^*(x) \tag{1}$$

where μ^* denotes the Haar measure on the ideles which is the product of the local Haar measures μ^* on $\mathbf{Q}_p^\times$ and $\mathbf{R}^\times$. This integral will not converge for a general $s \in \mathbf{C}$; the integrand isn't L^1. A sufficient condition for the integrand to be L^1 is that all the local integrands are L^1 and furthermore that the product of the local integrals is absolutely convergent. But we already worked these local integrals out! At the finite places we have

$$\int_{\mathbf{Q}_p^\times} f_p(x)|x|_p^s \mathrm{d}\mu^*(x)$$

which was $1/(1-p^{-s})$ for $Re(s) > 0$, and at the infinite place we get $\pi^{-s/2}\Gamma(s/2)$ again for $Re(s) > 0$.

It's a relatively straightforward check that if $\prod_p (1-p^{-s})^{-1}$ converges absolutely then this adelic integral converges, and for $\mathrm{Re}(s) > 1$ this will be the case because the product is just $\sum_{n\geq 1} n^{-s} = \zeta(s)$. So for $\mathrm{Re}(s) > 1$ the adelic integral above will converge, and it will converge to

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

This is exactly the function such that $\xi(s) = \xi(1-s)$ that we saw in the first lecture! Furthermore, replacing $\mathbf{Q}$ by more general number fields gives us Dedekind zeta functions of general number fields, and replacing some of the local factors with things like locally constant additive characters gives us Dirichlet L-functions and so on – all of them are adelic integrals like this and you can see some examples on the example sheets.

All we have to do now then, is to generalise the proof we gave of meromorphic continuation of ξ, and $\xi(s) = \xi(1-s)$, and we're done! The original proof I gave used Poisson Summation for $\mathbf{Z}$ in $\mathbf{R}$ (which followed from the classical Fourier inversion theorem) to prove a functional equation for the classical θ function $\theta(t) = \sum_{n\in\mathbf{Z}} e^{-\pi n^2 t^2}$, namely $\theta(1/t) = t\theta(t)$. We then showed $\xi(s) = \int_{t=0}^{\infty}(\theta(t)-1)t^{s-1}\mathrm{d}t$. We split this up as the sum of two integals:

$$\xi(s) = \int_{t=1}^{\infty}(\theta(t)-1)t^{s-1}\mathrm{d}t + \int_{t=0}^{1}(\theta(t)-1)t^{s-1}\mathrm{d}t$$

and substituted $u = 1/t$ into the second integral to get

$$\xi(s) = \int_{t=1}^{\infty}(\theta(t)-1)t^{s-1}\mathrm{d}t$$
$$+\int_{u=1}^{\infty}(\theta(1/u)-1)u^{-s-1}\mathrm{d}u.$$

Poisson summation applied to the second term gives $\theta(1/u) = u\theta(u)$ and tidying up shows that for $Re(s) > 1$ we have

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$$\xi(s) = \int_{t=1}^{\infty}(\theta(t)-1)t^{s-1}\mathrm{d}t$$
$$+\int_{u=1}^{\infty}(\theta(u)-1)u^{-s}\mathrm{d}u - 1/(1-s) - 1/s$$

and we were home and dry.

Let's now do all this adelically.

Chapter 6: The main theorem.

One of Tate's insights is that the correct analogue of the set $\mathbf{R}/\mathbf{Z}$ in this adelic context is the set $\mathbf{A}_k/k$. Let me run off a few things we know about the inclusion $\mathbf{Z} \to \mathbf{R}$. Firstly, $\mathbf{Z}$ is discrete, $\mathbf{R}$ is locally compact, $\hat{\mathbf{R}}$ (the Pontrjagin dual) is isomorphic to $\mathbf{R}$ again, and if we use the isomorphism $x \mapsto (y \mapsto e^{-2\pi ixy})$ to identify $\mathbf{R}$ with $\hat{\mathbf{R}}$ then we see that the annihilator $\mathbf{Z}^*$ of $\mathbf{Z}$ (that is, the elements $r \in \mathbf{R}$ such that $e^{-2\pi irn} = 1$ for all integers n) is just $\mathbf{Z}$ again. Hence the Pontrjagin dual of the discrete group $\mathbf{Z}$ is the compact group $\mathbf{R}/\mathbf{Z}$, and the dual of the exact sequence

$$0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to 0$$

is itself. Finally the action of $\mathbf{Z}$ on $\mathbf{R}$ admits a natural "fundamental domain" (that is, a subset D of $\mathbf{R}$, namely $[0, 1)$, with the property that the induced map $D \to \mathbf{R}/\mathbf{Z}$ is a bijection), and the measure of D, with respect to the standard Haar measure on $\mathbf{R}$, is 1.

You can guess what the analogue of most of these things are. The analogous short exact sequence is

$$0 \to k \to \mathbf{A}_k \to \mathbf{A}_k/k \to 0.$$

We have already seen that k is discrete in $\mathbf{A}_k$ and the quotient $\mathbf{A}_k/k$ is compact. It's not hard to check that the annihilator of k under the pairing $\mathbf{A}_k \times \mathbf{A}_k \to \mathbf{R}/\mathbf{Z}$ given by the sum of the Λs is again k, which shows that the Pontrjagin dual of the discrete group k is the compact group $\mathbf{A}_k/k$. We can even build D, an analogue of the fundamental domain above – it is just the product of $\mathbf{R}_P$ at all finite places, and a fundamental parallelopiped for the integers of k in $\mathbf{A}_{k,\infty}$ at the infinite places. The reason a D is useful in practice is that instead of integrating on $\mathbf{A}_k/k$ we can integrate on D instead. Amazingly, the additive measure of D is 1, because its value at the infinite place (the discriminant of k) is cancelled out by our choices of Haar measure at the finite places. The details of this calculation are in Tate's thesis, and the key input is that the discriminant can be calculated as a product of local terms.

Now we prove the analogue of the transformation property of the θ function. Instead of working with (the analogue of) $f(x) = e^{-\pi t^2 x^2}$ we set things up, for the time being at least, in more generality: we'll use a general function f for which we'll just assume everything converges.

First we observe that we have a natural Haar measure on the compact group $\mathbf{A}_k/k$: a function on $\mathbf{A}_k/k$ can be thought of as a "periodic" function on $\mathbf{A}_k$ (that is, one satisfying $f(x+\alpha) = f(x)$ for $\alpha \in k$) and, for a continuous function of this type, one checks easily that defining $\mu(f) = \int_D f(x)\mathrm{d}\mu(x)$ where μ is our fixed Haar measure on the adeles but *the integral is only over our fundamental domain* D, gives us a Haar measure on $\mathbf{A}_k/k$.

On the other side, if we endow k with the discrete topology then a natural Haar measure is just counting measure: a continuous function with compact support is just a function $f : k \to \mathbf{C}$ which vanishes away from a finite set, and we can define $\mu(f) = \sum_{\alpha \in k} f(\alpha)$. With these choices of Haar measure on k and $\mathbf{A}_k/k$, what is the constant in the Fourier inversion theorem? In other words, if we invert $F : k \to \mathbf{C}$ twice, we'll get $x \mapsto cF(-x)$. What is c? Unsurprisingly, it's 1 (as one can check by evaluating the transform of the transform of the characteristic function of $\{0\}$).

Reminder of Poisson Summation.

The Fourier inversion theorem on $\mathbf{R}/\mathbf{Z}$, when unravelled, just tells us the classical fact that if F is a continuous function on $\mathbf{R}/\mathbf{Z}$, viewed as a periodic function on $\mathbf{R}$, and if $a_m = \int_D F(x)e^{-2\pi imx}\mathrm{d}x$ is its mth Fourier coefficient, where $m \in \mathbf{Z}$ and $D = [0,1)$, and if $\sum_m |a_m|$ converges, then $F(x) = \sum_{m\in\mathbf{Z}} a_m e^{2\pi imx}$. We applied this very early on to a function $F(x)$ of the form $F(x) = \sum_{n\in\mathbf{Z}} f(x+n)$ where f was a function which was rapidly decreasing (it sent x to $e^{-\pi x^2 t^2}$), and we deduced

$$\sum_{n\in\mathbf{Z}} f(n) = F(0) = \sum_{m\in\mathbf{Z}} a_m.$$

And we computed a_m using this trick:

$$\begin{aligned} a_m &= \int_0^1 \sum_n f(x+n)e^{-2\pi imx}\mathrm{d}x \\ &= \int_0^1 \sum_n f(x+n)e^{-2\pi im(x+n)}\mathrm{d}x \\ &= \int_{\mathbf{R}} f(x)e^{-2\pi imx}\mathrm{d}x = \hat{f}(m) \end{aligned}$$

and so $\sum_{n\in\mathbf{Z}} f(n) = \sum_{m\in\mathbf{Z}} \hat{f}(m)$ as long as everything converges—this is the classical Poisson summation fomula.

Let's now do exactly the same thing, but on $\mathbf{A}_k/k$ instead of $\mathbf{R}/\mathbf{Z}$.

If $F \in L^1(\mathbf{A}_k/k)$ (that is, F is a function on the adeles and $F(x+\alpha) = F(x)$ for $\alpha \in k$, and furthermore if $\int_D F(x)\mathrm{d}\mu(x) < \infty$), then let's define $\hat{F} : k \to \mathbf{C}$ by

$$\hat{F}(\alpha) = \int_D F(x)e^{-2\pi i\Lambda(x\alpha)}\mathrm{d}\mu(x).$$

Lemma. With notation as above, if $\sum_{\alpha\in k} |\hat{F}(\alpha)|$ converges, then

$$F(x) = \sum_{\alpha\in k} \hat{F}(\alpha)e^{2\pi i\Lambda(\alpha x)}.$$

Proof. This is just the Fourier inversion theorem spelt out, together with the fact that $c = 1$. □

Corollary. $F(0) = \sum_{\alpha\in k} \hat{F}(\alpha)$. □

One last explicit definition: if $f \in L^1(\mathbf{A}_k)$ then, surprise surprise, define $\hat{f} : \mathbf{A}_k \to \mathbf{C}$ by $\hat{f}(y) = \int_{\mathbf{A}_k} f(x)e^{-2\pi i\Lambda(xy)}\mathrm{d}\mu(x)$, the usual Fourier transform, once we have identified $\mathbf{A}_k$ with its dual.

Theorem (Poisson summation, revisited.) If $f \in L^1(\mathbf{A}_k)$ is continuous, if $\sum_{\alpha\in k} f(x+\alpha)$ converges absolutely and uniformly for $x \in \mathbf{A}_k$, and if $\sum_{\alpha\in k} |\hat{f}(\alpha)|$ also converges, then

$$\sum_{\beta\in k} f(\beta) = \sum_{\alpha\in k} \hat{f}(\alpha).$$

Proof. (c.f. section 1.2.) Define $F : \mathbf{A}_k \to \mathbf{C}$ by $F(x) = \sum_{\beta\in k} f(x+\beta)$. Now by assumption the sum converges uniformly on $\mathbf{A}_k$, so F is continuous and periodic. Hence F, considered as a function on $\mathbf{A}_k/k$, is continuous with compact support and is hence L^1. Moreover, for $\alpha \in k$ we have (c.f. formula for a_m in 1.2)

$$\begin{aligned}
\hat{F}(\alpha) &= \int_D F(x) e^{-2\pi i \Lambda(\alpha x)} \mathrm{d}\mu(x) \\
&= \int_D \sum_{\beta \in k} f(x+\beta) e^{-2\pi i \Lambda(\alpha x)} \mathrm{d}\mu(x) \\
&= \sum_{\beta \in k} \int_D f(x+\beta) e^{-2\pi i \Lambda(\alpha x)} \mathrm{d}\mu(x) \\
&= \sum_{\beta \in k} \int_D f(x+\beta) e^{-2\pi i \Lambda(\alpha(x+\beta))} \mathrm{d}\mu(x) \\
&= \int_{\mathbf{A}_k} f(x) e^{-2\pi i \Lambda(\alpha x)} \mathrm{d}\mu(x) \\
&= \hat{f}(\alpha)
\end{aligned}$$

[where the interchange of sum and integral is OK because the sum converges uniformly on D, which has finite measure, and I've also used the fact (proved earlier) that $k \subset \ker(\Lambda)$, which I proved when showing $k = k^*$.] Hence

$$\begin{aligned}
\sum_{\beta \in k} f(\beta) &= F(0) \\
&= \sum_{\alpha \in k} \hat{F}(\alpha) \\
&= \sum_{\alpha \in k} \hat{f}(\alpha)
\end{aligned}$$

□

Next let's establish a global version of the "Re" function that we had on local quasi-characters.

Lemma. If $c : k^\times \backslash \mathbf{A}_k^\times \to \mathbf{R}_{>0}$ is a continuous group homomorphism, then $c = |.|^\sigma$ for some real number σ.

Proof. If J is the kernel of the global norm map, then $c(k^\times \backslash J)$ is a compact subgroup of $\mathbf{R}_{>0}$ and is hence $\{1\}$. So c factors through $\mathbf{A}_k^\times/J$ which, via the norm map, is $\mathbf{R}_{>0}$, and now taking logs we're done, because the only continuous group homomorphisms $\mathbf{R} \to \mathbf{R}$ are $x \mapsto \sigma x$.

Statement and proof of the main theorem.

Definitions. If $c : k^\times \backslash \mathbf{A}_k^\times \to \mathbf{C}^\times$ is a continuous group homomorphism then we say it's a *quasi-character* of $k^\times \backslash \mathbf{A}_k^\times$. We've just seen that $|c| : k^\times \backslash \mathbf{A}_k^\times \to \mathbf{R}_{>0}$ is of the form $x \mapsto |x|^\sigma$; define $\mathrm{Re}(c) = \sigma$. We let the set of quasi-characters of $k^\times \backslash \mathbf{A}_k^\times$ be a Riemann surface as in the local case, by letting the component of $c : k^\times \backslash \mathbf{A}_k^\times \to \mathbf{C}^\times$ be $\{c.|.|^s : s \in \mathbf{C}\}$. Note that in this case the Riemann surface is just an infinite union of copies of the complex numbers, indexed by the group $\hat{J}$ of characters of J. If c is a quasi-character of $k^\times \backslash \mathbf{A}_k^\times$ then let $\hat{c}$ be the character $x \mapsto |x|/c(x)$; note that $\mathrm{Re}(\hat{c}) = 1 - \mathrm{Re}(c)$.

Remark. I know very little about $\hat{J}$.

Recall that in the local setting we had a set Z consisting of "functions for which everything converged", and defined $\zeta(f,c)$ for $f \in Z$ and c a quasi-character with positive real part, as some sort of integral. Here's the analogy of this construction in the global setting.

Let Z denote the set of functions $f : \mathbf{A}_k \to \mathbf{C}$ satisfying the following "boundedness" conditions:

Firstly, we demand f is continuous and in $L^1(\mathbf{A}_k)$, and also that $\hat{f} : \mathbf{A}_k \to \mathbf{C}$ is continuous and in $L^1(\mathbf{A}_k)$.

Secondly (a condition that wasn't present in the local setting), we demand that for every $y \in \mathbf{A}_k^\times$, the sums $\sum_{\alpha\in k} f(y(x+\alpha))$ and $\sum_{\alpha\in k} \hat{f}(y(x+\alpha))$ converge absolutely, and moreover the convergence is "locally uniform" in the sense that it's uniform for $(x,y) \in D\times C$ for D our additive fundamental domain and C an arbitrary compact subset of $\mathbf{A}_k^\times$.

Thirdly, we demand that $f(y).|y|^\sigma : \mathbf{A}_k^\times \to \mathbf{C}^\times$ and $\hat{f}(y).|y|^\sigma$ are in $L^1(\mathbf{A}_k^\times)$ for all $\sigma > 1$ (note: this was $\sigma > 0$ in the local setting).

What are the reasons for these conditions? The first two mean that we can apply Poisson summation to f and indeed to the map $x \mapsto f(yx)$ for any $y \in \mathbf{A}_k^\times$. The local uniform convergence in the second condition is so that we can interchange a sum and an integral at a crucial moment. The third condition means that our global "multiplicative zeta integral" will converge for $\mathrm{Re}(s) > 1$.

Definition. If $f \in Z$ and $c : k^\times\backslash\mathbf{A}_k^\times \to \mathbf{C}^\times$ is a quasi-character with $\mathrm{Re}(c) > 1$, define

$$\zeta(f,c) = \int_{\mathbf{A}_k^\times} f(y)c(y)\mathrm{d}\mu^*(y)$$

(the Haar measure on $\mathbf{A}_k^\times$ being, of course, the product of our fixed Haar measures μ_v^* on $k_v^\times$).

The last condition in the definition of Z ensures the integral converges. Our main goal is:

Theorem. If $f \in Z$ then the function $\zeta(f,.)$ is holomorphic on the Riemann surface of quasi-characters c with $\mathrm{Re}(c) > 1$, and has a meromorphic continuation to all quasi-characters. Assume furthermore that $f(0) \neq 0$ and $\hat{f}(0) \neq 0$. Then $\zeta(f,.)$ has simple poles at the quasi-characters $c(x) = 1$ and $c(x) = |x|$, and no other poles (and \$1,000,000 attached to its zeros). Finally it satisfies the (very elegant!) functional equation

$$\zeta(f,c) = \zeta(\hat{f},\hat{c}).$$

We'll now start the proof of this, which of course is going to be a not-too-tough application of everything we have.

Before we prove the theorem let me make some definitions and prove some lemmas. We have $J \subseteq \mathbf{A}_k^\times$, the kernel of the norm function. Just as in the local case let's split this by finding $I \subset \mathbf{A}_k^\times$ isomorphic to $\mathbf{R}_{>0}$ such that $\mathbf{A}_k^\times = I \times J$. We do this by just choosing an infinite place $[\tau_0]$ of k and letting I be the copy of the positive reals in $k_{\tau_0}^\times$. We identify I with $\mathbf{R}_{>0}$ so that the norm map induces the identity $\mathbf{R}_{>0} \to \mathbf{R}_{>0}$, so if τ_0 happens to be a complex place then, because our complex norms aren't standard, what we're doing here is letting I be the positive reals in $\mathbf{C}^\times$ but letting the map $\mathbf{R}_{>0} \to I$ be $t \mapsto \sqrt{t}$.

For $f \in Z$ and $\mathrm{Re}(c) > 1$, we firstly break off this factor of I in the definition of the zeta integral: we write

$$\begin{aligned}\zeta(f,c) &= \int_{I\times J} f(y)c(y)\mathrm{d}\mu^*(y) \\ &= \int_{t=0}^{\infty}\int_{b\in J} f(tb)c(tb)\mathrm{d}\mu^*(b)\mathrm{d}t/t \\ &= \int_{t=0}^{\infty} \zeta_t(f,c)\mathrm{d}t/t\end{aligned}$$

where our measure on J is the one such that its product with $\mathrm{d}t/t$ on I gives us μ^* on $\mathbf{A}_k^\times$, and the last line is the definition of $\zeta_t(f,c) := \int_J f(tb)c(tb)\mathrm{d}\mu^*(b)$.

This is the analogue of $\xi(s) = \int_{t=0}^{\infty}(\theta(t)-1)t^{s-1}\mathrm{d}t$.

Let's think a little about

$$\zeta_t(f,c) = \int_J f(tb)c(tb)\mathrm{d}\mu^*(b).$$

We know that the integral defining $\zeta(f,c)$ converges, by assumption, for $\mathrm{Re}(c) > 1$, and hence the integrals defining $\zeta_t(f,c)$ will converge (at least for all t away from a set of measure zero). But these integrals are very docile: for $b \in J$ we have $|b| = 1$ by definition, so if $\mathrm{Re}(c) = \sigma$ then $|c(tb)| = |tb|^\sigma = t^\sigma$ is constant on J, and hence if the integral defining $\zeta_t(f,c)$ converges for one quasi-character c (which it almost always does) then it converges for all of them.

[$\zeta_t(f,c) = \int_J f(tb)c(tb)\mathrm{d}\mu^*(b)$.]

The problem, of course, is not in the convergence of the individual $\zeta_t(f,c)$; it's that as t goes to zero then $f(tb)$ will be approaching $f(0)$ and if this is non-zero, which it typically will be, then the integral of this function over the non-compact J might be getting very big, so $\int_{t=0}^{1}\zeta_t(f,c)\mathrm{d}t/t$ will probably diverge if, say, $\sigma < 0$ (because then t^σ is also getting big). This is the problem we have to solve.

Note also that we've written

$$\zeta(f,c) = \int_{t=0}^{\infty}\zeta_t(f,c)\mathrm{d}t/t$$

and that this is one of the crucial tricks. If $f = \prod_v f_v$ with f_v on k_v then we could compute the global integral as a product of local integrals—but in applications this would just tell us that our global zeta function is a product of local zeta functions, which will not help with the meromorphic continuation. The insight is to compute the integral in this second way. Note that Iwasawa independently had this insight in 1952.

Let's now see why ζ_t looks a bit like a theta function. Lets choose a fundamental domain E for $k^\times$ in J; then E is compact so has finite measure (the measure of E is related to the class number and regulator of k) and the integrals below are finite. Using $J = k^\times.E$ we get

$$\begin{aligned}\zeta_t(f,c) &= \sum_{\alpha\in k^\times}\int_{\alpha E} f(tb)c(tb)\mathrm{d}\mu^*(b)\\ &= \sum_{\alpha\in k^\times}\int_{E} f(t\alpha b)c(tb)\mathrm{d}\mu^*(b)\\ &= \int_E\left(\sum_{\alpha\in k^\times} f(t\alpha b)\right)c(tb)\mathrm{d}\mu^*(b)\end{aligned}$$

where the first equality is the definition, the second uses the fact that μ^* is a multiplicative Haar measure on J and that c is trivial on $k^\times$, and the third is an interchange of a sum and an integral which is justified by our rather strong uniform convergence assumptions on $f \in Z$ and the observation that the closure of E is a compact subset of $\mathbf{A}_k^\times$.

Exactly the same argument (changing f to $\hat{f} \in Z$, c to $\hat{c}$ and t to $1/t$) shows that blah $\zeta_{1/t}(\hat{f},\hat{c}) = \int_E\left(\sum_{\alpha\in k^\times}\hat{f}(\alpha b/t)\right)\hat{c}(b/t)\mathrm{d}\mu^*(b)$.

[$\zeta_t(f,c) = \int_E\left(\sum_{\alpha\in k^\times} f(t\alpha b)\right)c(tb)\mathrm{d}\mu^*(b)$]

Now that sum over $k^\times$ looks almost like a sum over k, but firstly the term $\alpha = 0$ is missing (so we'll have to add it in) and secondly we're not summing $f(\alpha)$ but $f(t\alpha b)$. So we'll have to work out what the Fourier transform of $x \mapsto f(txb)$ is. In other words, we need to see how the additive Fourier transform scales under multiplication. In the application of the lemma below we'll have $\rho = tb$.

Lemma. If $f : \mathbf{A}_k \to \mathbf{C}$ is continuous and in $L^1(\mathbf{A}_k)$, if $\rho \in \mathbf{A}_k^\times$ is fixed and if $g(x) := f(x\rho)$ then $\hat{g}(y) = \frac{1}{|\rho|}\hat{f}(y/\rho)$.

Proof. An elementary computation. We have

$$\hat{g}(y) = \int_{\mathbf{A}_k} f(x\rho)e^{-2\pi i\Lambda(xy)}\mathrm{d}\mu(x)$$

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and setting $x' = x\rho$ we have $\mathrm{d}\mu(x') = |\rho|\mathrm{d}\mu(x)$ and hence

$$\begin{aligned}\hat{g}(y) &= \int_{\mathbf{A}_k} f(x')e^{-2\pi i\Lambda(x'y/\rho)}\mathrm{d}\mu(x')/|\rho|\\ &= \frac{1}{|\rho|}\hat{f}(y/\rho)\end{aligned}$$

as required. □

So now let's add in the missing $\alpha = 0$ term to $\zeta_t(f,c)$, apply Poisson summation, and see what happens. Recall we just showed that $\zeta_t(f,c) = \int_E\left(\sum_{\alpha\in k^\times} f(t\alpha b)\right)c(tb)\mathrm{d}\mu^*(b)$ and that $\zeta_{1/t}(\hat{f},\hat{c}) = \int_E\left(\sum_{\alpha\in k^\times}\hat{f}(\alpha b/t)\right)\hat{c}(b/t)\mathrm{d}\mu^*(b)$.

Key Lemma. For an arbitrary $t > 0$ and c we have

$$\begin{aligned}&\zeta_t(f,c) + f(0)\int_E c(tb)\mathrm{d}\mu^*(b)\\ &= \zeta_{1/t}(\hat{f},\hat{c}) + \hat{f}(0)\int_E \hat{c}(b/t)\mathrm{d}\mu^*(b).\end{aligned}$$

Proof. As we've already remarked, the formulas we have just derived for $\zeta_t(f,c)$ and $\zeta_t(\hat{f},\hat{c})$ involve sums of $\alpha \in k^\times$.

The LHS of the lemma is hence what you get when you add the missing $\alpha = 0$ term: it's

$$\int_E \left(\sum_{\alpha\in k} f(t\alpha b)\right) c(tb)\mathrm{d}\mu^*(b) \qquad (1).$$

Similarly the RHS is

$$\int_E \left(\sum_{\alpha\in k} \hat{f}(\alpha b/t)\right) \hat{c}(b/t)\mathrm{d}\mu^*(b) \qquad (2).$$

So we need to show (1) = (2). The internal sum over k screams out for an application of Poisson summation, which, when applied to the function $x \mapsto f(txb)$ (we're allowed to apply Poisson summation because of our assumptions on f) gives

$$\sum_{\alpha\in k} f(t\alpha b) = \sum_{\alpha\in k} (\widehat{x \mapsto f(txb)})(\alpha) = \sum_{\alpha\in k} \frac{1}{|tb|}\hat{f}(\alpha/tb).$$

Hence formula (1) is equal to

$$\int_E \left(\sum_{\alpha\in k} \hat{f}(\alpha/tb)\right) c(tb)/|tb|\mathrm{d}\mu^*(b)$$

and now making the substitution $b \mapsto 1/b$, which doesn't change Haar measure, this becomes

$$\begin{aligned}&\int_E \left(\sum_{\alpha\in k} \hat{f}(\alpha b/t)\right) c(t/b)|b|/|t|\mathrm{d}\mu^*(b)\\ =&\int_E \left(\sum_{\alpha\in k} \hat{f}(\alpha b/t)\right) \hat{c}(b/t)\mathrm{d}\mu^*(b)\end{aligned}$$

which is (2)! This proves the lemma. □

We're finally ready to meromorphically continue our global zeta integrals. But before we do, let's try and figure out exactly what that fudge factor was that we had to add to $\zeta_t(f,c)$ to make that argument work in that last lemma: we added $f(0)$ times

$$\int_E c(tb)\mathrm{d}\mu^*(b).$$

What is this? Well if $c(x) = |x|^s$ is trivial on J then $c(tb) = t^s$ is constant for $b \in E$ (indeed, for $b \in J$), so the integral is just $t^s\mu^*(E)$, and one can check that the measure of E is $2^r(2\pi)^s hR/(w\sqrt{|d|})$—it's some finite non-zero number, anyway. But if c is non-trivial on J then, because it's always trivial on $k^\times$, it descends to a non-trivial character on the compact group $J/k^\times$"="E and the integral will hence be zero (distinct characters are orthogonal). So in fact we have

Corollary. If c is non-trivial on J and $f \in Z$ and $t > 0$ then $\zeta_t(f,c) = \zeta_{1/t}(\hat{f},\hat{c})$.

We're finally ready to prove the main theorem! I'll re-state it.

Theorem. If $f \in Z$ then the function $\zeta(f,.)$ is holomorphic on the Riemann surface of quasi-characters c with $\mathrm{Re}(c) > 1$, and has a meromorphic continuation to all quasi-characters. Assume furthermore that $f(0) \neq 0$ and $\hat{f}(0) \neq 0$. Then $\zeta(f,.)$ has simple poles at the quasi-characters $c(x) = 1$ and $c(x) = |x|$, and no other poles, (and \$1,000,000 attached to its zeros). Finally it satisfies the functional equation

$$\zeta(f,c) = \zeta(\hat{f},\hat{c}).$$

Proof. For $\mathrm{Re}(c) > 1$ the LHS zeta integral converges (by assumption on f) and is holomorphic in the c variable (differentiate under the integral). By definition, $\zeta(f,c) = \int_{t=0}^{\infty} \zeta_t(f,c)\mathrm{d}t/t$, which converges by assumption for $\mathrm{Re}(c) > 1$, and now we break the integral up into two parts:

$$\zeta(f,c) = \int_{t=1}^{\infty} \zeta_t(f,c)\mathrm{d}t/t + \int_{t=0}^{1} \zeta_t(f,c)\mathrm{d}t/t.$$

Now just as in the argument for the classical zeta function, I claim that the integral for $t \geq 1$ converges for all c, because the ideles tb showing up in the integral all have $|tb| = |t||b| = |t| \geq 1$ so if the integral converges for e.g. $\mathrm{Re}(c) = 2$ (which it does, by assumption, as $2 > 1$) then it converges for any c with $\mathrm{Re}(c) < 2$ (because the integrand is getting smaller).

That term isn't the problem. The problem term is the integral from 0 to 1, which typically only converges for $\mathrm{Re}(c) > 1$. So let's use the previous lemma, which has some content (Poisson summation) and see what happens. The simplest case is if $c(x) \neq |x|^s$ for any s (that is, c is non-trivial on J). In this case those extra fudge factors in the previous lemma disappear, and we see

$$\begin{aligned}\int_{t=0}^{1} \zeta_t(f,c)\mathrm{d}t/t &= \int_{t=0}^{1} \zeta_{1/t}(\hat{f},\hat{c})\mathrm{d}t/t \\ &= \int_{u=1}^{\infty} \zeta_u(\hat{f},\hat{c})\mathrm{d}u/u\end{aligned}$$

---

and this last integral also converges for all quasi-characters $k^\times \backslash \mathbf{A}_k^\times \to \mathbf{C}^\times$ because $u \geq 1$ so convergence again gets better as $\mathrm{Re}(c)$ gets smaller. Moreover our new expression for $\zeta(f,c)$, namely

$$\zeta(f,c) = \int_{t=1}^{\infty} \zeta_t(f,c)\mathrm{d}t/t + \int_{u=1}^{\infty} \zeta_u(\hat{f},\hat{c})\mathrm{d}u/u$$

converges for all c and makes it clear that $\zeta(f,c) = \zeta(\hat{f},\hat{c})$ (and that it's holomorphic for all c not in the component $|.|^s$). The proof is complete in this case!

We're not quite finished though: we need to deal with the component $c(x) = |x|^s$, where the argument is slightly messier because we pick up factors of $f(0)\int_E c(tb)\mathrm{d}\mu^*(b) = f(0)t^s\mu^*(E)$ and $\hat{f}(0)\int_E \hat{c}(\frac{1}{t}b)\mathrm{d}\mu^*(b)$. In this case (writing $c(x) = |x|^s$ now), the extra factors we'll see in the calculation will be (for $\mathrm{Re}(s) > 1$)

$$f(0)\mu^*(E)\int_{t=0}^{1} t^s \mathrm{d}t/t$$
$$= f(0)\mu^*(E)[t^s/s]_0^1 = f(0)\mu^*(E)/s$$

---

and

$$\int_{t=0}^{1}(\hat{f}(0)\int_E |b/t|^{1-s}\mathrm{d}\mu^*(b))\mathrm{d}t/t$$
$$= \hat{f}(0)\mu^*(E)\int_{t=0}^{1} t^{s-2}$$
$$= \hat{f}(0)\mu^*(E)[t^{s-1}/(s-1)]_0^1 = \hat{f}(0)\mu^*(E)/(s-1).$$

These functions (cst $/s$ and cst $/(s-1)$) clearly have a meromorphic continuation to $s \in \mathbf{C}$! So we have, for $c(x) = |x|^s$ with $\mathrm{Re}(s) > 1$,

$$\zeta(f,c) = \int_{t=1}^{\infty}\zeta_t(f,c)\mathrm{d}t/t + \int_{t=0}^{1}\zeta_t(f,c)\mathrm{d}t/t$$
$$= \int_{t=1}^{\infty}\zeta_t(f,c)\mathrm{d}t/t + \int_{u=1}^{\infty}\zeta_u(\hat{f},\hat{c})\mathrm{d}u/u$$
$$+ \mu^*(E)(-f(0)/s + \hat{f}(0)/(s-1))$$

and now we really have proved the theorem because this latter expression makes sense as a meromorphic function for all $s \in \mathbf{C}$, the integrals are all holomorphic for all $s \in \mathbf{C}$, and the expression is invariant under $(f,c) \mapsto (\hat{f},\hat{c})$. □

We've even computed the residues of $\zeta(f,|.|^s)$ at $s = 0$ and $s = 1$; they've come out in the wash.

They are $-f(0)\mu^*(E)$ and $\hat{f}(0)\mu^*(E)$ respectively. Recall that we computed $\mu^*(E) = 2^r(2\pi)^s \,\mathrm{Reg}_k\, h/w\sqrt{|d|}$.

Examples.

We have left open the logical possibility that $Z = \{0\}$, in which case our theory is empty. Let's check it isn't!

Example of a non-zero $f \in Z$: let's build $f : \mathbf{A}_k \to \mathbf{C}$ as a product of f_v. If v is finite let's just let f_v be the characteristic function of R_v. If v is infinite and real set $f_v(x) = e^{-\pi x^2}$ and if v is complex set $f_v(x+iy) = e^{-2\pi(x^2+y^2)}$. At the infinite places we've rigged it so $\hat{f}_v = \hat{f}$. At the finite places, $\hat{f}_v$ is $p^{-m/2}$ times the characteristic function of the inverse different of f, where p^m generates the discriminant ideal of k_v, so $\hat{f}_v = f_v$ at the unramified places but not at the ramified places.

We now have a problem in analysis: we need to check $f \in Z$. First let's check f and $\hat{f}$ are in $L^1(\mathbf{A}_k)$. Well, locally they are integrable, and at all but finitely many places the local integral is 1, so the infinite product trivially converges and gives the global integral.

Next let's check the third condition; we need to check that $f(y).|y|^\sigma$ is in $L^1(\mathbf{A}_f^\times)$ for $\sigma > 1$, and similarly for $\hat{f}$. Well the local factors are certainly in L^1—indeed, they are in L^1 for $\sigma > 0$, because we checked this when we were doing our local zeta integrals. But this isn't enough to check that the product is L^1: we need to check that the infinite product of the local integrals converges. We evaluated the local integrals at the finite places, when doing our local calculations, and they were $(1-p^{-\sigma})^{-1}$ for $k = \mathbf{Q}$ (I did these in class) and more generally $p^{-m/2}(1-q^{-\sigma})^{-1}$ if k is a finite extension of $\mathbf{Q}$ and we're doing the computation at a P-adic place,

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with residue field of size q and discriminant ideal (p^m) (I mentioned these on the example sheet; the proof is no more difficult). So we need to check that $\prod_P(1 - N(P)^{-\sigma})^{-1}$ converges for $\sigma > 1$—and it does; this is precisely the statement that the zeta function of a number field converges for $\mathrm{Re}(s) > 1$, which is proved by reducing to $k = \mathbf{Q}$ and then using standard estimates. This argument applies to both f and $\hat{f}$, which are the same away from a finite set of places.

Finally we have to check the second condition (the one that let us apply Poisson summation and interchange a sum and an integral). Let y be a fixed idele, let x be a fixed adele, and let's first consider

$$\sum_{\alpha \in k} f(y(x+\alpha)).$$

First I claim that this sum converges absolutely. Because look at the support of f: at the finite places it's supported only on "integral ideles" $\mathbf{A}_k^f \cap \prod_{v<\infty} R_v$, so,

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whatever y and x are, $f(y(x+\alpha))$ will actually equal zero if, at any place, the denominator of αy beats the denominator of xy. So this sum, ostensibly over all of $k^\times$, is really only over a fractional ideal in k, and now convergence is trivial because at the infinite places (and there is at least one infinite place) f is exponentially decreasing, and there are only finitely many lattice points with norm at most a given constant.

Now why is the convergence locally uniform? It's for the same reason. If y and x vary in a compact then the fractional ideal above might move but for compactness reasons the lattice won't get arbitrarily small (it's not difficult to write down a formal proof) and it's hence easy to uniformly bound the sums involved.

So the main theorem applies! What does it say in this case?

Well, $\zeta(f, |.|^s)$ and $\zeta(\hat{f}, |.|^{1-s})$ are closely related to, but not quite, the zeta function of k. Indeed if we write S_∞ for the infinite places of k and S_f for the finite places which are ramified in $k/\mathbf{Q}$ then $\zeta(f, |.|^s) = \prod_v \zeta(f_v, |.|^s)$ (the right hand integrals are local zeta integrals), which expands to

$$\prod_{v \in S_\infty} \zeta(f_v, |.|^s) \prod_{v \in S_f} (Nv)^{-m_v/2} \prod_P (1 - N(P)^{-s})^{-1}$$

and $\prod_{v|\infty} \zeta_v(f_v, |.|^s)$ is a load of gamma factors—exactly the fudge factors which you multiply $\zeta_k(s) = \prod_P (1 - N(P)^{-s})^{-1}$ by to get (definition) $\xi_k(s)$. So $\zeta(f, |.|^s) = \xi_k(s)|d|^{-1/2}$ with d the discriminant of k. Now $\zeta(\hat{f}, |.|^{1-s})$ is almost the same, except that $\hat{f} \neq f$ at the finite ramified places: the local integral of f_v at the finite place is easily checked to be $p^{-ms}/(1-q^{s-1})$, so for $\mathrm{Re}(1-s) > 1$ we have $\zeta(\hat{f}, |.|^{1-s}) = \xi_k(1-s)|d|^{-s}$ and we deduce

$$\xi_k(1-s) = |d|^{s-1/2}\xi_k(s).$$

Slightly better: if we set

$$Z_k(s) = \xi_k(s).|d|^{s/2} = \zeta_k(s).\prod_{v|\infty}\zeta(f_v, s).|d|^{s/2}$$

then we get

$$Z_k(1-s) = Z_k(s).$$

This is the functional equation for the "Dedekind zeta function", that is, the zeta function of a number field.

Moreover, we know that the pole at $s = 1$ of $\zeta(f, |.|^s)$ is simple with residue $\hat{f}(0)\mu^*(E) = \hat{f}(0)2^r(2\pi)^s \operatorname{Reg}_k .h/(w\sqrt{|d|})$, and $\hat{f}(0) = |d|^{-1/2}$, so the pole at $s = 1$ of $\xi_k(s) = \zeta(f, |.|^s)|d|^{1/2}$ has residue $2^r(2\pi)^s \operatorname{Reg}_k .h/(w\sqrt{|d|})$. Moreover the local zeta factors at the real infinite places are $\pi^{-s/2}\Gamma(s/2)$ which equals 1 at $s = 1$, and at the complex infinite places are $(2\pi)^{1-s}\Gamma(s)$ which is again 1 at $s = 1$, so we deduce

$$\lim_{s\to 1}(s-1)\zeta_k(s) = 2^r(2\pi)^s \operatorname{Reg}_k .h/(w\sqrt{|d|})$$

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which is called the analytic class number formula and which is used crucially in both analytic arguments about densities of primes and in algebraic arguments in Iwasawa theory.

THE END